Eliminating infinity

For anyone who is uncomfortable with references to 'infinity' in mathematics, this article offers simple alternatives to the word 'infinity' and to the symbol for infinity, ∞. It also provides some historical background to the topic and insights into the underlying logical issues.

Is this article relevant to you? It depends on what you believe about numbers.

Do you believe in infinity?

People hold different basic beliefs about numbers and some are compatible with an absolute, actual infinity while others are not. These four questions will indicate if you are someone who might prefer to eliminate uses of the word 'infinity' and its symbols (e.g. $\infty, \aleph, \omega$), as I do.

1. Do you think there is a number so big that it is the biggest and no number is bigger? It's a number you cannot add 1 to and make a bigger number. (Here we are talking about numbers in principle, not about numbers we currently can represent on paper or with a computer.)
2. Do you think there is a number so big that there is no number less than it to which you could add 1 and reach the number? Somehow it is out of reach in that sense.
3. Do you think that it is meaningful to talk about the total number of elements in an infinite set or infinite sequence of numbers, even though they are endless? (Here again we are talking about sets and sequences that are infinite in principle, so in this question it doesn't matter whether they could exist in practice.)
4. Do you think it is meaningful to say that one infinite sequence is longer than another infinite sequence, even though both are endless?

If you answered 'no' to all or most of these questions, as I did, then you are probably not someone who believes in what is often called 'actual infinity'.

You may not have noticed before, but some of the notation currently taught in schools and universities makes references to infinity as if actual infinity exists. This can lull us into thinking that actual infinity exists even if our basic beliefs, when considered directly and explicitly, conflict with that.

For me, this is a matter of mental hygiene. I don't want to have misleading words and symbols in my mind. I also don't want to endorse ideas that I think are wrong, even though it is usually only a matter of principle.

If you don't believe in actual infinity and would like to learn more about how to eliminate unintended references to it, then please read on.

Alternative words and notation

The main objective here is to avoid using words, phrases, or symbols that suggest – even subtly – that there is a number, or something like it, that is the biggest possible number, bigger than all others, a sort of Last Chance Saloon for the numbers.

The ∞ symbol is used often to express ranges of Real numbers that are not limited in one or both directions. For example $[0, \infty)$ is the set of real numbers that, in set builder notation, would be expressed as:

$$\{x : \mathbb{R} \mid x \geq 0\}$$

and pronounced as 'for all Real numbers greater than or equal to zero.'

The alternative to using the symbol for infinity in this situation is just to use the set builder notation instead. Since the range is unlimited at the top there is simply no need to mention the top of the range at all. The same principle applies to ranges with no lower limit.

The ∞ symbol also appears in expressions of the sum of an infinite series. For example:

$$\sum_{r=1}^{\infty} \frac{1}{r^2}$$

can be rewritten slightly as:

$$\sum_{r \in \mathbb{N}_1} \frac{1}{r^2}$$

pronounced as 'the sum for all Naturals from 1 of …'

Set builder notation can be used here too to express more complicated ranges for the series.

A similar situation exists for definite integrals.

$$\int_a^{\infty} f(x)dx$$

becomes,

$$\int_{x \geq a} f(x)dx$$

while

$$\int_{-\infty}^{\infty} f(x)dx$$

becomes simply,

$$\int_{x \in \mathbb{R}} f(x)dx$$

The idea is simple: if there are no limits then don't try to state some.

With limits of functions there are also easy ways to drop the ∞ symbol.

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

becomes

$$\lim_{x \to} \frac{1}{x} = 0$$

or,

$$\lim_{x \uparrow} \frac{1}{x} = 0$$

Pronounced 'the limit as x rises without end…'

Similarly,

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

becomes

$$\lim_{\leftarrow x} \frac{1}{x} = 0$$

or,

$$\lim_{x \downarrow} \frac{1}{x} = 0$$

Pronounced as 'the limit as x falls without end …'

Where there is no limit, this should be stated clearly. For example:

$$\lim_{0 \leftarrow x} \frac{1}{x} \; does \; not \; exist$$

Similarly, never talk about the number of Natural numbers overall because the Naturals are an infinite set. Never talk about the number of Rational or Real numbers, even in any finite interval, because, again, those are infinite sets.

Some finer points

Eliminating the appearance of infinity in your mathematical writing highlights some points that are usually implicit.

Convergent series

It is widely understood and often written that:

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

But what does the summing notation mean when the ∞ symbol is used? The situation is clear in an example without the ∞ symbol, such as

$$\sum_{k=1}^{3} \frac{1}{2^k} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3}$$

but it would be wrong to say that,

$$\sum_{k=1}^{\infty}\frac{1}{2^k}=\frac{1}{2^1}+\frac{1}{2^2}+\cdots+\frac{1}{2^{\infty-1}}+\frac{1}{2^\infty}$$

because ∞ is not a number.

It would also be wrong to say that the sum in the infinite case is the value we get if we keep on adding more terms in an unlimited way.

Mathematical induction can be used to show that the sum calculated in this way would not reach 1. At each stage in such a calculation the sum so far will be

$$\sum_{k=1}^{N}\frac{1}{2^k}$$

When N is 1, the sum is 0.5. That means there is a gap between the sum and 1. This is the base case.

The effect of adding another term is always to halve the gap. For any other value of N, if there is a gap between the sum at that stage and 1, then there will be a gap at the next stage, the sum to $N+1$.

So, there is a gap between the sum and 1 when $N=1$ and whenever there is a gap there is a gap when one more term is added, and so there is a gap for all Natural numbers[1]. The sum never actually reaches 1.

This argument also clarifies that adding more and more terms to the series does not lead us to some final value. Each further addition changes the value so far, just a tiny bit.

(In this argument, as with all reasoning about the infinite and infinitesimal, the numbers we think about are not always numbers that we can write down or represent on a computer. The numbers might be too big, or too small, or too accurate to be represented with an existing number writing system. The numbers we are thinking about are theoretical, idealized numbers. Perhaps we could write them if we invented a suitable system or a bigger computer. This is traditional in mathematics.)

[1] Proponents of infinity have an argument to get around this type of proof that involves asserting that there are numbers so big that they don't have a number that is one less than them. This blocks the inductive argument – but of course only if you agree that such numbers exist.

The right way to understand the summing notation when infinity is involved is as a limit. The limit idea is usually implicit but can be made explicit:

$$\lim_{n\uparrow}\Big(\sum_{k\in 1..n}\frac{1}{2^k}\Big)=1$$

Being a limit means that the sum of the series to n gets closer and closer to 1 as n rises, so that no matter how close to 1 you want it to be there is a value for n that gets you even closer. There is no other value that is the limit in this sense.

What it does not necessarily mean is that the sum of the series actually arrives at 1 if you increase n far enough – 'to infinity'.

We could abbreviate the 'limit' notation in these situations implicitly or explicitly. The implicit style might look like this:

$$\sum_{k\,\in\,\mathbb{N}_1}\frac{1}{2^k}=1$$

Here, the fact that a limit is involved is not made clear and you have to know that when an unlimited set like the Natural numbers is involved then a limit is implied.

An explicit abbreviation style could use something other than the usual '=' sign. A number of symbols could be used. The symbol '~' is sometimes used to mean 'asymptotic', while arrows (→ and perhaps also ←) are often used to mean 'approaches', and 'approaches the limit' is sometimes written as '≐'. The style might look like this:

$$\sum_{k\,\in\,\mathbb{N}_1}\frac{1}{2^k}\sim 1$$

The same issue of implicit limits arises when a series is written by listing the first few terms then writing '…' to indicate endless continuation in the same pattern.

Again, the special symbol could be used to highlight the use of a limit. Typical applications are the binomial theorem with fractional powers, Taylor expansions, and Maclaurin expansions. For example,

$$e\sim 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$$

and

$$\cos[x] \sim 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

The idea of these being asymptotic rather than exactly equal agrees nicely to our experience of using series like these to compute numbers. They give us an algorithm that can be continued as far as necessary to calculate the decimal representation of the number to as many decimal places as we want.

Recurring decimals

Recurring decimals are another situation where a limit is implicit.

Which is true?

$$0.999999 \ldots = 1 \; or \; 0.999999 \ldots \neq 1$$

Alternatively written as:

$$0.\dot{9} = 1 \; or \; 0.\dot{9} \neq 1$$

Majority published opinion today is that the equality is true, justified by an implicit limit indicated by the two dot notations (the '…' and the dots over digits).

Without this the equality would not be true. The gap argument used above that employs mathematical induction would show that adding another digit never eliminates the gap between the value so far and 1, which is the limit. The sum would also fail to settle to a single final value as further terms are added.

The use of the limit could be made explicit using the full limit notation or with the 'asymptotic' symbol or something like it. For example:

$1 \sim 0.\dot{9}$

$\frac{1}{2} = 0.5$

$\frac{1}{3} \sim 0.\dot{3}$

$\frac{1}{4} = 0.25$

$\frac{1}{5} = 0.2$

$\frac{1}{6} \sim 0.1\dot{6}$

$\frac{1}{7} \sim 0.\dot{1}4285\dot{7}$

$\frac{1}{8} = 0.125$

$\frac{1}{9} \sim 0.\dot{1}$

$\frac{1}{10} = 0.1$

Comparing numerosity

The number of numbers in a finite set is a meaningful concept. The number of numbers in an infinite set is not. There is no number that represents the number of numbers in an infinite set. Some care over language is needed to avoid suggesting that there is.

We cannot say:

'The number of Natural numbers is infinity.'

We can say:

'The set of Natural numbers is an infinite set.'

'There are infinitely many Natural numbers.'

Less obviously, we cannot say:

'The number of Rational numbers is greater than the number of Natural numbers.'

'There are more Rational numbers than Natural numbers.'

We can say:

'The Natural numbers are a subset of the Integers.'

'The even Natural numbers are less dense than the Natural numbers.'

Since it is meaningless to talk about the number of elements in an infinite set we cannot compare the number of elements in infinite sets[2].

However, there are alternative approaches to comparisons.

One is that we can try to compare the density of numbers in comparable intervals. A number of ways to do this have been devised for subsets of the Natural numbers, including Natural Density (also known as Asymptotic Density or Arithmetic Density), the Schnirelmann Density, and the Logarithmic Density.

Here is a simple and rather limited example using an idea of my own.

I suggest doing it using the relative number of numbers within an interval carefully chosen to allow a fair comparison.

This can be illustrated using the example of Naturals versus even Naturals. We need to choose an interval that starts where the two sets have a common element and ends at a higher point where they have a common

[2] This did not stop Georg Cantor from constructing an elaborate system that supposedly does just this. Not surprisingly, it leads to some strange conclusions.

element. However, it needs to include the lower end but exclude the upper end.

In symbols, let $\mathbb{N}$ be the set of Natural numbers and $\mathbb{N}_e$ be the set of even Natural numbers, and let a and b be elements in both sets, such that $a < b$, then the two subsets whose cardinality we want to compare are:

$$\{x : \mathbb{N} \mid a \leq x \land x < b\}$$

and

$$\{x : \mathbb{N}_e \mid a \leq x \land x < b\}.$$

The relative number of elements within each in that interval is calculated as a fraction, like this:

$$r = \frac{\#\{x : \mathbb{N} \mid a \leq x \land x < b\}}{\#\{x : \mathbb{N}_e \mid a \leq x \land x < b\}}$$

For example, if $a = 0$ and $b = 10$, then the relative density in that interval is:

$$r = \frac{\#\{x : \mathbb{N} \mid 0 \leq x \land x < 10\}}{\#\{x : \mathbb{N}_e \mid 0 \leq x \land x < 10\}}$$

$$= \frac{10}{5}$$

$$= 2$$

In this example the ratio will always be 2 for all choices of a and b such that:

$$a \in \mathbb{N} \land a \in \mathbb{N}_e \land b \in \mathbb{N} \land b \in \mathbb{N}_e \land a < b$$

However, in other examples this is not necessarily going to be the case so a general definition of this idea of relative density would have to be for specific intervals.

Also, not all number sets can be compared in this way since they have to have some common elements and there are also sets where the ratio is not defined. For example, the ratio between Real numbers and Integers is infinite, as is the ratio between the Rational numbers and Integers.

Rational numbers and Real numbers are infinitely dense, so comparisons of their densities are not possible. We know they are infinitely dense because, in between any two distinct Rational numbers there lies at least one more Rational number, their average. The same applies to Real numbers. Irrational numbers are also infinitely dense because, between any pair of distinct Rational numbers lies at least one Irrational number, being the number that divides the difference between the two Rationals in the ratio $\sqrt{2}:2-\sqrt{2}$.

Another way to compare infinite sets of numbers is by looking for where one contains the other. For example, all Natural numbers are Integers, but not all Integers are Natural numbers. In that sense, the Integers are a 'bigger' set, even though we cannot put a number on the size of either set. Similarly, one can argue that Rational numbers are a strict subset of the Reals.

A perspective on number systems

My personal preferred perspective on infinite number systems is that they are an idealization that is helpful in the design and assessment of practical number systems.

In this perspective, numbers are not the same thing as the labels we apply to them, such as '10', '101101', and 'A6'.

Number systems should be rooted in shared, everyday experiences. These are experiences that are universal and understandable to all. We use numbers to observe and communicate 'how many' and 'how much', and this distinction is reflected even in the grammar of many of our natural languages. They reflect our two uses of numbers, which are counting and measuring. Neither is more fundamental than the other.

Numbers for counting

The **Natural** numbers are an idealized system for counting objects. You could imagine counting pebbles as you put them into a box, or sheep as they enter a field.

The Natural numbers are idealized in that there are no gaps that would prevent us from counting particular collections and because they go on forever upwards. This idealized system never runs out of numbers, and in that sense alone is infinite. The idealization makes it possible to reason about the results an ideal number system should be able to produce, without the inconvenience of having to check for going out of range at every stage. It does not necessarily reflect an infinite reality.

These abstract, idealized numbers are unique and sequenced yet have no names.

Practical implementations of number systems for counting, such as binary representations using 16 bits on a digital computer, or decimal notation written on paper with a pen, give permanent labels to the Natural numbers, but only to a few of them. Examples of such labels in three different systems are 5 (decimal), 101 (binary), and V (Roman).

Modern practical number systems are cleverly designed so that the labels are not just arbitrary names, like 'George' or 'Bob', but are instead structured strings of symbols that we can compute with. For example, you might know from memory that 2+3=5, but asked for 23 x 28 you would probably use a method based on the digits used to label the numbers to work out the answer, 644.

These permanent labels are not like the variables names that mathematicians introduce when talking about arbitrary or as-yet-unknown numbers. For example, when a mathematician writes 'Let m and n be Natural numbers such that…' the m and n are temporary names, not permanent labels for the numbers. Perhaps later the mathematician will deduce the permanent labels that go with those two names and they will be '3' and '6'.

The Natural numbers have an obvious starting point for counting, but no end point.

In practical number systems for counting we usually label a subset of the Naturals, starting at the starting point with 0 (or 1 depending on your preference) and labelling all the naturals from there upwards until the labels run out. For example, with 16 binary digits that usually means starting at zero and running out at 65,535 (in decimal notation).

For numbers written on paper with a pen there is no hard end to the labels we can write, but it does quickly become impractical to write very long numbers by hand or even to print them on paper so, in practice, we do not continue endlessly. (Perhaps there is a case for idealized decimals and practical decimals.)

Numbers for relative counting

The **Integers** provide an idealized system for counting objects relative to a starting point. You could imagine having a box that already contains many pebbles but then counting pebbles going out of it (subtract 1 each time) and pebbles coming in (add 1 each time) starting from zero. By the end of your session the pebbles in the box might have reduced, giving you a negative count, or increased, giving a positive count, or stayed the same, leaving you on zero.

The Integers are idealized in that they also have no gaps and they go on forever both up and down. They never run out. This time there is no natural starting point for counting, so a starting point has to be part of the idealized design.

These abstract, idealized numbers are unique and sequenced yet have no names.

Again, practical implementations of relative counting numbers give labels, but only to a few of the idealized Integers. With binary digits, one of those digits might be used to say if the number is positive or negative, with the other digits being used for magnitude. Again, it is usual to give a label to the starting point (e.g. zero) and to all the Integers around it, until the labels run out.

Numbers for measuring

The **Real** numbers provide an idealized system for measuring quantities, with direction. They are idealized in that they go on without end both up and down, and because there are no gaps at all. We can imagine measuring using an idealized line (the edge of something perfectly straight and endlessly long) where every point on that line has a Real number associated with it. These points have no width at all. They are locations, not areas or intervals.

To say there are no gaps means that, wherever on an idealized number line you might happen to point (with the edge of an idealized pointer) there is a Real number to go with that point. I think this captures our intuition about there being a Real number for every magnitude.

In contrast, practical implementations of measuring numbers give labels to only a few of those Reals, and there are gaps all over the place. Not only is it not possible to label Reals because they are too high or too low, but it is also impossible to label every Real in even a tiny interval.

What we do instead is to use a wide variety of designs to spread out the labelled points in useful ways. For example, floating point representations are good for very small numbers or very large ones, but not for adding very small to very large. Do that and the very small numbers get lost when rounding off.

Some systems aim to spread the labelled points so that the gaps between them increase as you get further from zero, perhaps even keeping the percentage rounding errors roughly constant.

Another design involves representing numbers not by a simple label, but by an algorithm capable of generating a decimal representation of the number accurate to any desired (but practical) number of digits. Easy examples are fractions, such as $^1/_3$ generating as many 3s as you like in the decimal expansion. Less obvious examples are roots, values from trigonometrical functions, e, and π, all of which can be calculated from converging series. Alan Turing provided a list of types of number that are 'computable' like this.

There are also notations that label very, very large numbers, such as towers of exponentials, Knuth's arrows, and Bowers' operators. This far out, the gaps between labelled points are also gigantic.

In addition to having a designated starting point for measuring ('zero') as with the Integers, the Reals are also usually given a second point that represents a single unit away from the starting point (what we would usually call 0 to 1).

In most labelling systems for Real numbers the labels we use for Natural numbers are reused to label Integer multiples of the unit Real number.

The ability to reason about Real numbers mathematically provides a way to assess the performance of practical implementations of measuring numbers. For example, if you know from mathematics with the idealized Reals that the result of a calculation should be a particular value, and a computer using practical numbers gives a result that is a few percent out, then you know that the computed numbers are wrong. The compromises made in designing the practical number system and operations on those numbers have perhaps led to unacceptable inaccuracy.

This perspective on number systems and the role of idealized, infinite number systems is very different from the famous, published ideas that you can read about today. These famous formulations date from about a century ago and typically reflect a desire to write axioms (i.e. assumptions) from which all other properties of numbers and basic operations on them can be deduced. These were to take mathematics away from intuition and the real world and make it a purely logical endeavour (except for the initial assumptions).

There was also a long held view that counting was somehow more fundamental than measuring and that, therefore, what was needed was a way to define measuring numbers using only counting numbers. The logical gymnastics used in the attempt are elaborate, highly artificial, and hard to follow.

I think it is clear that the Naturals, Integers, and Reals are idealizations (though you may not agree with me on exactly how) and that our numbering systems are limited in comparison.

Interlinking number sets

Having said that we have invented number systems for counting and for measuring, how can we account for the fact that we often regard Natural numbers as a subset of the Integers, which in turn are a subset of the Rationals, which are a subset of the Reals.

The link between the measuring and counting numbers is a matter of design since we choose to define a unit length and assign the counting numbers to the unit lengths. In practical number systems we ensure that labels match up. For example, 2 corresponds to two unit lengths, which might also be written as 2.00.

Alternative beliefs

If, like me, you answered 'no' to the questions at the start of this article then you might be puzzled that anyone at all answers 'yes' to them.

Surprisingly, the main published position in mathematics today is consistent with 'yes' to most if not all. It is frequently stated, without reference to limits, that:

$$0.\dot{9} = 1$$

and that

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

and that the number of Natural numbers is the same as the number of even Natural Numbers.

It wasn't always like this but Georg Cantor was successful, eventually, in building a following for his ideas to such an extent that many believe them without question today, even if they know nothing of the background.

Cantor was a German mathematician who lived from 1845 to 1918. He believed that God was infinite (in the sense that Cantor understood infinity) and that his theories about infinite sets were God's ideas, revealed to Cantor by God.

Cantor believed that you could assert anything you liked in mathematics as long it was proven from accepted assumptions and no contradictions appeared.

Cantor was driven to promote his ideas to mathematicians and theologians, even writing to the Pope. Despite opposition from many and recurrent bouts of depression that put him in hospital, he toiled on, eventually winning some supporters.

The number of people in the world who fully understand Cantor's arguments must be quite small because there are many papers, written in German, and they are extremely complicated and obscure. The style of argument is typical of mathematics of the time, with long arguments devoid of any sense of direction, and a deluge of terminology.

Cantor's position probably gained an advantage by comparison with alternatives at the time. The idea of using logic to deduce conclusions in mathematics was a fresh and attractive one, with some of Cantor's opponents preferring intuition.

I suspect that Cantor's ideas would have been forgotten quickly if he had faced opposition from just one alternative position, backed by a group of people in agreement and with status, that relied on logic but started with assumptions that were acceptable to most people.

The war over his ideas went on for decades and we can see the residual bitterness and irrationality of that war even today in books and internet forum postings. People unhappy with the 'establishment' position sometimes angrily describe it as moronic, while the 'establishment' describes their critics as cranks. This is a debate that is making no progress.

Problems with recurring decimals

One battleground is recurring decimals. The assertion that $0.\dot{9} = 1$ is often seen as something that many people have 'trouble accepting'. There have even been studies published exploring the cognitive problems that this causes for people and trying to find ways to convince people more effectively that the equality is true.

These attempts to convince rarely mention the implicit use of limits, which would be correct and should be convincing. This may be because it is very hard to find a definition of the recurring decimals notation that explicitly mentions the use of limits.

Instead, attempted proofs typically use algebra to try to show that the assertion is true without mentioning the implicit use of limits.

What is striking about such claimed proofs that $0.\dot{9} = 1$ is the contrast between the confident tone of the writer and the ease with which the flaws can be seen.

One argument starts with the assertion that:

$$\frac{1}{3} = 0.33333 \ldots$$

Multiply both sides by 3 and you get:

$$1 = 0.99999 \ldots$$

The problem here is that it starts by assuming something that is almost the same as what is being 'proved'.

The gap argument would show that (without using a limit):

$$\frac{1}{3} \neq 0.33333 \ldots$$

and that, in general, rational fractions are not exactly equal in value to the infinitely recurring decimals they generate. So, the argument starts from a false assertion (if not using limits).

Another argument for the equality is that the sum from adding in each additional digit 9 converges to a limit of 1 and, therefore, its sum is equal to 1.

But limits and equality are not the same thing and it is possible for a series to converge to a limit without actually reaching it.

Yet another argument for equality says that:

$$1 - 0.\dot{9} = 0.\dot{0}$$

And that:

$$0.\dot{0} = 0$$

So there is no difference between 1 and $0.\dot{9}$ and, therefore, they must be equal.

Again, the inequality is assumed away with the initial assumption. Start instead with the finite equality:

$$1 - 0.9 = 0.1$$

Take this a step further:

$$1 - 0.99 = 0.01$$

Keep on adding '9' digits without limit. The difference on the right continues to exist but of course gets smaller and smaller.

This is in fact the gap argument in another form and demonstrates the opposite conclusion i.e. inequality.

One last example of an argument that is, at least, controversial, goes like this. Let $x = 0.\dot{9}$. Multiply both sides of this little equation by 10 to get:

$$10x = 9.\dot{9}$$

Subtract the first equation from the second to get:

$$9x = 9$$

And, finally, divide both sides by 9 to arrive at:

$$x = 1$$

Again, the argument starts with a dodgy assumption (if no implicit limits are involved). If instead the same process is followed for ever increasing 9s the conclusion is different.

Let $x = 0.9$ and then multiply both sides by 10.

$$10x = 9$$

Subtract the first equation from the second, giving:

$$9x = 8.1$$

Divide through by 9:

$$x = 0.9$$

Repeat this for ever increasing numbers of decimal digits and at no point will magic occur. The letter x stubbornly refuses to change its form or value.

The lack of explicit references to limits either in the notation or in common explanations of the notation has caused a great deal of confusion. The omission probably occurred because many people thought that limits were not necessary because a value is reached when the digits of a decimal number are continued 'to infinity.'

To show that:

$$0.\dot{9} = 1$$

it is only necessary to explain that the notation means the limit of the sequence 0.9, 0.99, 0.999, etc is 1, and then prove that 1 is indeed the limit of this sequence. In outline, this involves showing that the difference between the sequence so far and 1 can be made as small as you like by adding enough digits, and that adding even more digits will not make the gap larger again. This is obvious when you know that the gap gets cut by 90% when each new digit is added.

Limit ordinals

The idea of a limit ordinal illustrates the style of argumentation at the heart of the long-running war.

An ordinal is a number used to express an element's place in a sequence. A limit ordinal is supposedly one of these numbers that is so big that somehow it has no predecessor. That is, there is no other number that you can add one to and get the ordinal limit.

Even more bizarrely, having asserted the existence of limit ordinals, Cantor went on to say that it was perfectly reasonable to add 1 to a limit ordinal.

This is a very odd idea, but one obvious reason for proposing it strongly is that it fends off arguments by mathematical induction (such as the gap argument above) that otherwise would have been devastating.

Cardinality of infinite sets

The arguments around the so-called 'cardinality' of infinite sets illustrate another argument tactic at the heart of the infinity war.

The word 'cardinality' applied to finite sets means the number of elements in the set. This cannot work for infinite sets because there is no number that can be used for their size. Cardinality in the usual sense is not definable for sets of unlimited size.

The only way to talk about 'cardinality' for infinite sets is to apply the word to a different idea when talking about those infinite sets.

Changing the meaning of a word is something that humans do quite often and it can lead to confusion and reasoning mistakes.

Two finite sets are said to have equal cardinality if they both have the same number of elements. For example:

$$\#\{1,2,3,4,5\} = 5$$

$$\#\{2,4,6,8,10\} = 5$$

$$\#\{1,2,3,4,5\} = \#\{2,4,6,8,10\}$$

For infinite sets a different idea has become established, thanks to Cantor. In this case, two sets are said to have equal cardinality if there is a bijection between them. This is a function that links the elements one-to-one with no gaps.

This is true for finite sets. For example,

$$\{1 \mapsto 2, 2 \mapsto 4, 3 \mapsto 6, 4 \mapsto 8, 5 \mapsto 10\}$$

is a bijection that compares and equates the two sets above.

Cantor's new idea was to apply the same criterion to infinite sets. For example:

$$\{1 \mapsto 2, 2 \mapsto 4, 3 \mapsto 6, 4 \mapsto 8, \ldots\}$$

This definition of equal cardinality leads to some odd sounding conclusions. For example, mathematicians today typically say that the cardinality of the set of Natural numbers is the same as the cardinality of the set of even Natural numbers. The reason is, as shown above, that there is a bijection between them. This bijection can be written as a rule and using the rule you can find the Natural number that goes with each even Natural number, and vice versa.

Intuitively it seems obvious that there should be twice as many Naturals as even Naturals if any such statement can be made at all. What is going on here and how can we get around this problem?

The issue is still simply that cardinality in the usual sense does not apply to infinite sets so the comparison is meaningless.

However, if we want something that agrees with our intuition then we can look at the density of numbers within a set relative to the Natural numbers.

The ambiguity of 'countable'

The word 'countable' is another one whose shifting meaning has confused the arguments about infinite sets.

The ambiguity of the word 'countable' comes about because, in ordinary conversation, we tend to act as if something can be counted completely if we can identify discrete units that we can start counting. We do this because what we are counting is usually finite and so if we can start counting then we can imagine finishing the count, although it may take too long to be practical. For example, sheep can be counted. Money can be counted. If asked whether atoms can be counted we might say that, in theory, they could be but not in practice unless you have a very small sample or you are happy with an approximate answer.

When we come to talk about counting sets of items whose size is unlimited then this breaks down. We can identify discrete items and start counting them but we cannot count them all, even in principle. They are 'countable' in the sense that we can start counting them, but not 'countable' in the sense that we can completely count them.

The sense in which 'countable' is currently used in mathematics is that something is 'countable' if we can start counting, even if we could never finish. That is not the same as 'countable' in ordinary conversation.

Conclusion

It's not necessary to refer to infinity if you don't want to, and if your beliefs are inconsistent with an 'actual infinity' you may prefer to go that way.

The origins of some of today's best known ideas about infinity are surprisingly mystical and that is, perhaps, another reason for adopting infinity-free writing and thinking habits.

Appendix: Density of numbers

Density of Real numbers

Consider any two Real numbers that are not equal:

$$r_1 > r_2$$

The mean of the two will be between them and not equal to either. It will also be a Real number because there are no gaps on the Real number line.

So, for any two Real numbers that are not equal there is at least one Real number that lies between them. We can always squeeze one more in.

The unlimited supply of Real numbers can be looked at in another way. Consider the interval $[0,1)$. Using the decimal notation the number of numbers in this range that can be represented with just one digit after the decimal point is 10. The set is:

$$\{0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$$

The number of numbers that can represented using two digits after the decimal point is 100, and in general the number of numbers that can be represented with d digits after the decimal point is 10^d.

As d increases, the number of numbers that can be represented rises very rapidly and if d increases without limit then so does the number of numbers that can be represented.

Not all the Reals can be represented exactly by decimal numbers, so the number of Reals is even more than that suggested by the argument based on decimal numbers.

Density of Rational numbers

Consider any two Rational numbers that are not equal:

$$\frac{p_1}{q_1} < \frac{p_2}{q_2}$$

The mean of the two will be between them and is given by:

$$\frac{\frac{p_1}{q_1} + \frac{p_2}{q_2}}{2}$$

which is,

$$\frac{p_1 q_2 + p_2 q_1}{2 q_1 q_2}$$

and also a rational number because both numerator and denominator are integers.

So, for any non-equal rational numbers there is at least one other rational number that lies between them.

Alternatively, consider for example just the interval $[0,1]$. We can imagine beginning to list the rational numbers in this interval as follows:

$$\frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \frac{0}{4}, \dots$$

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

$$\frac{2}{2}, \frac{2}{3}, \frac{2}{4}, \frac{2}{5}, \dots$$

$$\frac{3}{3}, \frac{3}{4}, \frac{3}{5}, \frac{3}{6}, \dots$$

$$\dots \dots \dots$$

This is an infinite sequence of infinite sequences. Doing the same for another interval of the same length would involve just adding an integer to each of these numbers, so $[0,1]$ serves as an adequate example.

Consequently, the relative density between Rational numbers and Integers is infinite.

Density of Irrational numbers

Rational and Real numbers are frustratingly endless. Not only is there at least one rational number between every pair of rational numbers, but there are also at least two irrational numbers between any two rational numbers. Instead of interpolating with the average of the two rationals, split that gap $\sqrt{2}: 2 - \sqrt{2}$ and $2 - \sqrt{2}: \sqrt{2}$ and you get two irrationals in that gap.

This perhaps creates a problem for Dedekind's idea of finding gaps by splitting the Rationals using irrational dividing points. You might find a gap, but does that mean there is one irrational number there or two? The least upper bound property is shared by the integers, and by the integers divided by 10, divided by 100, and so on.