Eliminating imaginary numbers

If you've ever been given a hard time for grumbling about imaginary numbers (or complex numbers more generally) then this article is for you.

In contrast, if you've ever tried to correct someone with misgivings about imaginary numbers (which you perhaps prefer to call 'complex' numbers), then you will find this article uncomfortable or annoying reading. Nevertheless, please read on and reconsider your position.

Take a look around the internet now for articles and postings about complex numbers and you will get an impression that contrasts with the historical reality. The strong impression you will get is that complex numbers are vital, convenient, and that anyone who has a problem with them is mentally limited or just doesn't understand yet. The idea is that complex numbers seem strange initially but they are so wonderful we should ignore our misgivings and use them anyway.

The historical reality is that the high point for complex numbers was reached about 150 years ago. The more recent development of vector methods has led to a gradual elimination of complex numbers that continues to this day, but now very slowly.

This article discusses:

- the main reason we should eliminate complex numbers completely;
- actions we should take;
- the progress made already;
- responses to some of the common arguments given in favour of continuing with complex numbers; and
- some suggestions for alternative notation.

The main problem with i

The idea of a number that is the square root of -1 is bizarre but the main problem with i is its cultural impact.

Considering just the numbers we are familiar with for counting and measuring in the real world, there is no number that, multiplied by itself, gives a negative result. Multiply two negative numbers and you get a positive number.

If you apply mathematics to a practical problem, such as optimizing a machine, or designing a safe building, you might find you need to solve an equation. If it turns out that no real number solves the equation then it's back to the drawing board because there is no physical way your idea can be made, consistent with your equation.

To take a simple geometrical example, if you want a rectangular roof to have a perimeter of 40m and an area of 80m^2 then there are two possibilities (to 3 s.f.): 14.4m by 5.23m, or 5.23m by 14.4m. There are two because working this out requires solving a quadratic equation. However, if you want the roof to have a perimeter of 40m but an area of 110m^2 then you are out of luck because there is no way to do better than an area of 100m^2 – not in the real world using rectangles[1].

The idea of i is to pretend that negative numbers do have square roots, call the square root of -1 i, and proceed to do maths with it as if nothing unusual is going on.

Continuing with the example of a rectangular roof with a perimeter of 40m but an area of 110m^2, this produces two answers: $10 + i\sqrt{10}$

[1] A regular pentagon gets very close and there is an ellipse that exactly meets the requirements.

m by $10 - i\sqrt{10}$ m and $10 - i\sqrt{10}$ m by $10 + i\sqrt{10}$ m. But, of course, these answers have no meaning or value in the real world and a roof with these dimensions cannot be built.

The inherently illogical act of assuming the existence of something impossible, then making deductions with it, causes most people to be a bit concerned when it is first encountered. It took mathematicians almost 300 years to persuade themselves that it was a safe and acceptable procedure, despite the obvious logical problem.

More recent mathematical inventions mean that this leap of faith has not been needed for over 100 years, but some use of i continues, and is vigorously defended, even today.

This logical issue raised by using i has important *cultural* consequences. These are what matters from a practical point of view.

The continued use of i sends a message to students of mathematics and to others who hear about it: reasoning with absurdities is acceptable in mathematics. This is not a helpful message to teach and it undermines the credibility of mathematics and of mathematicians, and the credibility of physics and physicists.

A number of the arguments often used in favour of continuing to use i are themselves obviously flawed. This, combined with the sometimes patronising or even sneering way they are delivered, sends an even more damaging message: reasoning with absurdities is desirable in mathematics and not doing so is a sign of ignorance and mental limitations.

The fact that using these impossible numbers in the way they are used does not lead to incorrect answers, even in the real world, does not solve the cultural problem.

Although public objections to i are currently rare, there have been periods in history when they were strongly maintained, and even today there is a gentle movement towards other methods.

Policies to consider

The cultural damage caused by the continued use and promotion of imaginary numbers can be reduced if, overall, there is:

- a reduction in the use of non-logical justifications for continued use and teaching of complex number methods and notation;
- greater acceptance of, and more understanding responses to, people who question complex numbers;
- more frequent acknowledgement that the real reason for continued use of complex numbers is inertia and slow progress of reform;
- consistent preference in science, engineering, and mathematics for methods that do not use imaginary numbers – typically using simple trigonometric functions or vectors instead;
- increased development and use of efficient, modern vector methods, especially in areas where this has been slow to happen such as quantum mechanics;
- increased teaching of modern vector methods; and
- reduced teaching of complex number methods, ultimately leaving them as a historical topic.

This is not something that is going to be driven by politicians or law makers. It's more likely to be the result of literally millions of individual, personal decisions.

To press home the case for decisions in favour of logical alternatives to complex numbers the following sections provide:

- a historical perspective on the current situation with complex numbers;
- responses to common arguments promoting continued use and teaching of complex numbers; and
- proposals for an alternative notation free from i.

The rise and fall of complex numbers

The history of complex numbers helps to explain why they were, for a long period, so popular and why they have lost popularity in more recent times.

Solutions of cubic equations

It took something like 300 years for mathematicians (some of them at least) to persuade themselves that imaginary numbers had some kind of validity or usefulness.

(The historical facts in this section come from the Hyperjeff network.)

In 1484 Nicolas Chuquet wrote that some equations led to imaginary solutions but dismissed them.

Similarly, in 1545 Girolamo Cardano showed that solutions to some polynomials led to square roots of negative numbers but called them 'sophistic' and 'useless'.

However, in 1572 Rafael Bombelli published his 'wild idea' that one could sometimes reach real, valid solutions to equations via square roots of negative numbers. It took him over 20 years to get around to publishing this idea.

Also in a more supportive way, in 1629 Albert Girard published a book in which he retained the imaginary roots in order to give general rules that connected an equation with its roots. (This may be the first example of preferring this concise result over a more useful and truthful one.)

However, in 1637 Rene Descartes used the term 'imaginary' for the first time and saw them as a sign that no solution existed.

In 1670, Gottfried Wilhelm Leibniz suggested that imaginary numbers were somehow half way between existing and not existing.

Then, in 1673, John Wallis suggested an early way to represent complex numbers geometrically.

Subsequently, there seem to have been more determined efforts to develop consistent theories about imaginary numbers.

In 1714 Roger Cotes deduced the formula, $-i\varphi = ln(\cos\varphi - i\sin\varphi)$, though few people noticed.

The much more famous Leonhard Euler had more of an impact. In 1747 he showed a way to define the logarithm of a negative number in imaginary terms. Then, in 1748, he showed the famous result that $e^{i\theta} = \cos\theta + i\sin\theta$, though he was not the first to deduce this. In 1749 he showed that a complex number to the power of a complex number is also a complex number.

Also in 1749, Jean le Rond d'Alembert constructed functions of a complex variable and obtained what were later called the Cauchy-Riemann equations, defining differentiability of functions of a complex variable.

In 1777 Euler finally invented the symbol i. It had taken almost 300 years to reach this point but now imaginary numbers had a kind of respectability, at least for some of the mathematicians we remember.

Key points may have been the discovery that correct, real solutions could be found despite a detour through the imaginaries, and the idea of a geometric interpretation of the complex numbers that helped connect them with real applications.

Almost like vectors

Most students today can see that complex numbers are sort of like vectors, and in the days before vectors and vector operations had been properly developed these complex numbers must have seemed quite exciting.

Caspar Wessel, in 1797, published the style of diagram we are used to today, using the 'y' axis as the imaginary axis, but few people noticed at the time, perhaps because he was Danish.

In 1806 Jean Robert Argand had more luck with his graphical approach, and in the same year Adrien-Quentin Buée produced something similar.

Various mathematicians made progress with calculus in the 'complex plane'.

In 1831 Carl Friedrich Gauss published his theory of complex numbers (calling them 'complex' for the first time), with a geometric interpretation and a rigorous construction of the algebra. Gauss introduced the term 'complex' because he thought 'imaginary' had the wrong connotations and had made the whole area seem unnecessarily mystical. This all helped with respectability.

William Rowan Hamilton, in 1833, produced a formal algebra of real number couples that mirrored the algebra of complex numbers. This gave further respectability to complex numbers, though it might just as well have been used to show that they were unnecessary.

In 1835 Hamilton published again on the same theme, linking his pairs with (x, y) coordinates.

For a long time, Hamilton had been trying to find a way to extend the 2-dimensional complex numbers into 3 dimensions, presumably so that they could be used to model things in 3 dimensional space. He could not do it.

However, on October 16th, 1843, he had the idea of quaternions – tuples of 4 numbers interpreted as corresponding to points in 3 dimensional space. The algebra of these called for even more square roots of negative numbers than the complex numbers. Hamilton was already a well-respected and influential person and immediately began promoting this idea. He wrote two books on the subject, explaining their algebra and geometry.

A supporter of Hamilton's called Peter Guthrie Tait played an important role. In the 1850s he started applying quaternions to topics in physics, such as magnetism and electricity, and his approach produced strong reactions among scientists.

This was the high point for imaginary numbers, but it began to produce a slow rebellion in favour of the new, simpler vector methods that avoided references to impossible numbers altogether.

The impact of vectors

The origins of vectors are ancient but properly worked out methods as we see today were not widely known until long *after* imaginary numbers had become established.

(The historical facts for this section come from John Labute's useful page.)

In 1827 August Ferdinand Mobius had published a short book on directed line segments with applications that looked very much like vectors, but he was more interested in other things and nobody took much notice.

In 1844 Hermann Grassmann published a book that expanded the idea of vectors from 2 or 3 dimensions to n dimensions. He also played with ideas similar to modern matrix and linear algebra, vector, and tensor analysis. Unfortunately, it was abstract, with a complicated notation and no examples. Also, Grassman was just a secondary school teacher and did not have the position to promote his ideas effectively.

Despite these false starts, resistance to quaternions and a preference for vectors began to show.

Benjamin Peirce was the most famous mathematician in the USA and wrote praising Hamilton and his work on quaternions. Quaternions had been his favourite subject. And yet, in Peirce's massive book, *A System of Analytical Mechanics* (1855), he excluded quaternions and in his later book *Linear Associative Algebra* (1870) he wrote instead about his 'algebra of space'.

James Clark Maxwell had studied with Tait in Edinburgh and Cambridge and was also a Scot. He wrote positive points about quaternions but in reality was suggesting a purely vectorial approach, without the imaginary numbers.

However, the really effective advocacy and development of vectors over quaternions took place in the USA.

J Willard Gibbs, perhaps most famous for his work on thermodynamics, wrote notes on vector analysis for his students at Yale University. This began in 1881 and the notes were also widely distributed in the USA and Europe. The notes were printed as the first book on modern vector analysis in 1901, more than 120 years after Euler introduced the symbol i.

Gibbs had studied both Hamilton's quaternions and Grassman's obscure work on vectors but decided in favour of vectors for his work in physics.

There seems to have been a battle between the supporters of quaternions and the supporters of the new vector methods.

In the 1890s and first decade of the 20th century, Tait and others attacked vectors and defended quaternions. At the same time, many scientists and mathematicians began developing their own vector methods. (Competition between advocates of vector methods may have helped quaternions linger on longer than they otherwise would have.)

One particularly successful advocate of vectors was Oliver Heaviside, an English physicist. In a three volume work published in 1893, 1899, and 1912) he attacked quaternions and developed his own vector analysis. Maxwell's equations for electromagnetism were revised by Heaviside into the simpler vector notation used today.

Tait seems to have been annoyed by this and sent letters to Nature attacking Heaviside's methods. Heaviside did not shy away from controversy and in his old age he became increasingly bitter and eccentric.

I wonder if it is the emotion of this old battle that we can see lingering on in today's defences of complex numbers.

The advance of vectors continues. Henry G Booker wrote a book called *A vector approach to oscillations* in which he developed a system of rotating vectors exactly analogous to complex numbers, without references to i. This was published in 1965 but has made little impact so far.

Modern survival and alternatives

Today, complex numbers are still taught in universities and still advocated by some.

They linger on in physics and engineering where sinusoidal waves or motion are involved, though even here there is (nearly) always a published alternative approach that is free of imaginary numbers.

As an example, voltage and current in alternating current circuits containing resistors, capacitors, and inductors can be represented in two ways.

Without imaginary numbers the voltage and current are, respectively:

$$v[t] = V_{max}\cos[\omega t]$$

$$i[t] = I_{max}\cos[\omega t - \varphi]$$

The complex alternative looks like this:

$$v[t] = V_{max}e^{i\omega t}$$

$$i[t] = I_{max}e^{i(\omega t - \varphi)}$$

but expands to,

$$v[t] = V_{max}(\cos[\omega t] + i\sin[\omega t])$$

$$i[t] = I_{max}(\cos[\omega t - \varphi] + i\sin[\omega t - \varphi])$$

showing that the real part matches the plainer version and the imaginary part adds no information. Working out the power provided by alternating current using the complex approach requires taking just the Real part of the expressions involved.

There are alternative formulae for calculating the overall impedance of the circuit with or without the use of complex numbers.

Another example is the Fourier series, which represents periodic functions (e.g. a musical note) as the sum of a series of trigonometrical functions. This can be written with or without i.

In addition to the complex version,

$$s[x] = \sum_{n\in\mathbb{Z}} c_n\, e^{i\frac{2\pi nx}{P}}$$

there are two purely-Real alternatives:

$$s[x] = \frac{A_0}{2} + \sum_{n\in\mathbb{N}} A_n \sin\left[\frac{2\pi nx}{P} + \phi_n\right]$$

and

$$s[x] = \frac{a_0}{2} + \sum_{n\in\mathbb{N}} a_n \cos\left[\frac{2\pi nx}{P}\right] + b_n \sin\left[\frac{2\pi nx}{P}\right]$$

These equations are all talking about the amplitude, frequency, and phase shift of a series of sinusoidal waves. There's nothing inherently imaginary about any of that. The purely-Real forms make that plain as well as excluding the impossible i.

One famous area of physics where complex methods still have a virtual stranglehold is quantum mechanics. Although vector alternatives exist they are not promoted strongly at present and the dominant approach is to use imaginary numbers. Some even claim this is essential but this cannot be true. Hamilton showed, long ago, that a system of algebra with the same outward behaviour can be defined that lacks references to i.

Perhaps this branch of physics has been protected from change by its conceptual difficulty and the apparent preference for weirdness. Few people want to take on the task of rewriting it. Exceptions include Kadin (2005), Barrand (2014), and Kwong (2014).

Teaching using ordered pairs

A number of modern authors teach complex numbers by starting with a definition in terms of ordered pairs and operations using them, with no mention, initially, of i.

Some explain their motivation for doing this, which is usually to help students who they believe find complex numbers difficult to accept or to understand. (I suspect many critical thinking students have difficulty understanding why their teacher is advocating the bizarre idea of i.)

The explanation begins with the idea that the true, proper definition of complex numbers is in terms of ordered pairs, written as

$$(a, b),$$

and then adds two operations, one for adding two of these pairs and another for multiplying them. These mimic the effect of adding and multiplying complex numbers. Division and subtraction are derived from these.

The argument then often proceeds to show that any Real number, a, can be matched with a pair $(a, 0)$. Furthermore, the effect of addition and multiplication with the new operators mimics addition and multiplication of their matching Reals.

The letter i is then introduced as a name for a particular pair, $(0,1)$, which has the property that

$$(0,1) \times (0,1) = (-1,0)$$

Complex number format is then introduced, explaining that a Real number, a, can be thought of as an abbreviation for $(a, 0)$ and a number multiplied by i, such as ia, can be seen as $(0,1) \times a$, which is $(0, a)$. So, in summary, a complex number like $a + ib$ is really (a, b), and that's the proper, formal way to look at complex numbers.

The strength of this approach is that it makes a lot of good points. The weakness of the approach is that it uses these good points to try to legitimize i by saying that it's not really what it looks like or what many people say it is.

Common arguments for complex numbers

This section reviews and counters some arguments for use of imaginary numbers. Not all the arguments are mathematical or logical and this has added to the cultural problem caused by continued use and promotion of imaginary numbers.

Historical patterns

New types of number

One of the most common arguments for using i is just a ploy to reduce critical thinking about i. The ploy says that objections were raised to both irrational and negative numbers when they were first proposed, but these have been resolved, so don't worry about imaginary numbers because whatever worries you might have either have been resolved or will go away once you understand the topic better.

Clearly, the lack of logical problems with negative and irrational numbers is not proof that imaginary numbers are free of logical problems. It is damaging to the culture of reason that should dominate mathematics to argue in this way.

Solutions to equations

Another argument based on historical patterns goes like this: Real numbers are solutions to equations, so anything that is a solution of an equation is a number too, and just as valid. The same logic might argue that ice cream is something I like so anything I like is ice cream. Not so.

A variation on this is to say 'If we want to have solutions to these equations then we need to expand our concept of numbers.' The argument then proceeds on the assumption that we do want to do that and it's sufficient justification for what is then done. Clearly, we do not necessarily want to have solutions for equations that do not have usable solutions, and even if we did that would not necessarily be sufficient justification for doing something illogical.

Relativism

The relativism argument says that mathematics involves choosing a set of axioms and deducing their implications. The choice of axioms is arbitrary – who is to say what is really 'true', and what is 'truth' anyway? The rules of complex numbers lead to no contradictions and so they are valid mathematics.

But axioms are not an arbitrary choice. To get practical value from mathematics and real insight into the world we live in we need to choose axioms that are true. What is true can, ultimately, be established by tests with the real world.

The better way to use mathematics is to choose axioms that are true, and obviously so, and deduce what we can from just those. We try to avoid axioms that feel a bit dodgy.

Making assumptions that are clearly not true is the last thing we should be doing and that means we should avoid i.

Convenience

Another common argument for using i is that it is 'convenient'. Again, this is damaging to the culture of reason because convenience is not evidence of correctness.

Number of roots of a polynomial

Various examples of convenience are offered. One such is that the number of solutions a polynomial equation has can be stated more briefly if we allow i. With i in play we can say that the number of solutions a polynomial of degree n has is n. The trouble is that this is not a useful piece of information because, in reality, polynomial equations representing anything in the real world will not necessarily have n solutions.

For example, quadratic equations of the form:

$$ax^2 + bx + c = 0$$

have two solutions, one solution, or no solutions, depending on the values of a, b, and c, if we exclude i. That's the reality and if you are using mathematics to think about something in the real world, such as the thickness of girders in a bridge, you only want to know about the solutions that will really exist.

Truth beats brevity.

Use as rotating vectors

Complex numbers often look good as a compact way to write about waves. Instead of writing Cartesian coordinates,

$$(R\cos\theta\,, R\sin\theta)$$

we can write,

$$Re^{i\theta}$$

which looks smaller.

However, convenience in mathematics is mostly about how notation and theorems have been developed. If we reserve the most compact notations for the things we want to write most often, and develop identities and other theorems to speed up deductions we often want to make, then convenience is developed.

For example, the polar coordinates,

$$(R,\theta)$$

are also compact.

It is not really $\sqrt{-1}$

The presentation of complex numbers as being 'formally' ordered pairs with operations associated, says that there is an interpretation of i that does not involve the impossible $\sqrt{-1}$, and therefore it is acceptable to write i. In other words, 'When I said X I was not wrong, because when I say X I actually mean Y.'

If the use of ordered pairs is really the proper understanding then the notation should be that of ordered pairs i.e. (a, b). Using $a + ib$ allows the notion of i as $\sqrt{-1}$ to continue unreformed.

They work

One of the most compelling things about complex numbers must have been, and still is, the fact that they seem to work. Correct real solutions to cubic equations can be calculated despite a detour through imaginary space. Correct identities involving trigonometric functions can be derived. And so on.

This seems like a very strong case in favour of using i and in favour of i being something that exists in some sense.

However, the work on ordered pairs shows that the approach can succeed just as well without i.

This idea of an alternative system based on two dimensional vectors and a suite of vector operations is expanded in detail in Appendix A. The usual vector operations are augmented by a new set based on a form of vector multiplication that simply matches the effect of complex multiplication.

This even has a physical interpretation, in terms of rotating position vectors. Booker's vector approach to oscillation shows that complex numbers really model rotating vectors in a way that makes perfect sense in the real world. This is most obvious when you see complex numbers on an Argand diagram and see the impact of multiplication of two complex numbers. They look just like two vectors starting from the origin.

To illustrate this idea, consider the equation:

$$x^2 = -1$$

written as an equation that, apparently, involves Real numbers. In the world of complex numbers the x is interpreted as complex and the solutions to this equation are i and $-i$.

If we use rotating vectors instead and make that explicit using polar coordinates and other notation tweaks then the equation can be written as:

$$(R, \theta)^{(2,0)} = (-1,0)$$

The solution to this involves using operations on the vectors as follows:

$$(R^2, 2\theta) = (1, \pi)$$

so it can be seen that the two unique solutions are the rotating vectors (in polar form),

$$\left(1, \frac{\pi}{2}\right) \text{ and } \left(-1, \frac{\pi}{2}\right)$$

which are analogous to i and $-i$. There is no mystery.

Necessity in a special situation

The much more technical argument that started the whole imaginary numbers movement is this one: we need i to write a 'closed form' general statement of the solutions of a cubic polynomial in terms of surds, therefore we need i.

You have to read this carefully to understand how narrow the problem is to which i provides a solution.

We can find all the real-world solutions of cubic equations without i to any desired level of practical accuracy using numerical methods, so i is not needed for solving cubic equations.

There is a trigonometric formula that works where the cubic has three real roots and a similar hyperbolic formula that covers the other possibility, which is where there is just one real root. That means we can find the roots with analytical formulae.

There are also some cases where the solution of a cubic equation can be written in 'closed form' without i. There just remains the pesky situation that sometimes arises where, if you want to write the solution in a neat 'closed form' way, and surds are involved, then there is, supposedly, no way to do it without including square roots of negative numbers.

But there is at least one way to do it without including square roots of negative numbers. If a cubic equation is written with the Real unknown replaced by a vector unknown, and the addition, multiplication, and exponent operations are defined appropriately (as in Appendix A), then sometimes solutions will be found that have the form $(r, 0)$. When that happens we know that the matching cubic equation in a Real unknown has a real solution, r. It does not matter if reaching that solution involved going through rotating vectors of other forms.

Other ploys

Other ploys used to defend imaginary numbers are designed to lessen resistance. They include:

- saying or implying that people who object are ignorant or closed-minded;
- saying (as Gauss did) that there is nothing odd or mysterious about the square root of -1 and that it's only the inappropriate name 'imaginary' that creates that impression;
- glamorizing the ability to tolerate working with illogical, absurd ideas as a mark of intellectual maturity;
- claiming they are essential for quantum mechanics;

- claiming their use is established, traditional, and popular; and
- retelling the history of complex numbers without including the history of vectors and their impact on use of complex numbers.

Notation proposals

Many applications where complex numbers are still used by some today can be tackled by simply using trigonometric functions. Often the imaginary part adds no information.

This is probably the main and the best way to avoid using imaginary numbers. However, there are some niches where complex numbers cannot be avoided as easily.

To remove complex numbers from these requires a notation that does everything they do, clearly and concisely, but without references to i or to complex numbers.

Appendices A, B, and C explain and demonstrate the use of a notation for this purpose. It is based on the established idea of ordered pairs, but just develops the practical details a little further than usual.

The refinements include the following:

- The pairs are usually thought of as position vectors.
- Cartesian and polar forms are visually distinct due to a suffix after the brackets e.g. $(2,4)_c$, and $\left(3,\frac{\pi}{2}\right)_p$.
- There is frequent conversion between Cartesian and polar forms to give the most succinct formulae in each case.
- As well as addition, subtraction, multiplication, and division there are exponents, logarithms, and even some trig functions.
- The operators and functions are given symbols/names that identify them as taking ordered pairs as input and being consistent with the specially defined multiplication.
- All of these mimic the effect of the analogous object for complex numbers.
- All of these mimic the effect of Real number operations when the vector's second value is zero.
- i does not appear at all.

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Appendix A: Analogous notation

Purpose

The objective of this appendix is to show one way to add the functionality of complex numbers to two dimensional vectors, and so show how complex numbers can be replaced even in the niches where they currently seem to be the only option.

Design ideas used

The main thing that needs to be added to the usual toolkit for two dimensional vectors is a rotating form of multiplication that replaces multiplication of complex numbers. This is indicated by the symbol $\odot$.

However, this also leads to a need for other operations on the vectors that are consistent with this rotating form of multiplication, such as for raising to a power ($\circledast$), and functions such as for logarithms ($\log_\circ$ and $\ln_\circ$) and trigonometric functions ($\cos_\circ, \sin_\circ$).

Also, when working with complex numbers it is common to switch between two forms frequently. These forms are the $a + ib$ form and the $Re^{i\theta}$ form. One reason for this is that some formulae are more concisely expressed using the $a + ib$ form, while others are more easily expressed using the $Re^{i\theta}$ form, and still others are easiest in a mixed form.

To provide similar convenience when working with two dimensional vectors, it helps to have different looking notations for Cartesian and polar forms – $(a, b)_c$ and $(R, \theta)_p$ respectively.

Basic notation

Here are the analogous notations, with the distinction between Cartesian and polar forms shown:

With i	**Without i**
$a + ib$	$(a, b)_c$ (c for Cartesian)
$Re^{i\theta}$	$(R, \theta)_p$ (p for polar)
	$\mathbb{V} == \mathbb{R} \times \mathbb{R} == \mathbb{R}^2$
$z: \mathbb{C}$	$r: \mathbb{V}$

The type $\mathbb{V}$ is just an abbreviation of two real numbers in a pair, designed to suggest that the pair is being used as a vector.

Extractions

In complex numbers the modulus and argument are often referred to. These relate to the angle and length used in the polar form.

The real and imaginary parts correspond to the horizontal and vertical components in the Cartesian form but it makes no sense to refer to real and imaginary parts when talking about plain vectors.

With i

$\forall a, b: \mathbb{R} \cdot$

$$\mathrm{Re}(a + ib) = a$$

$$\mathrm{Im}(a + ib) = b$$

$$|a + ib| = \sqrt{a^2 + b^2}$$

$$\arg(a + ib) = \tan^{-1}\frac{b}{a}$$

$\forall R, \theta: \mathbb{R} \cdot$

$$\mathrm{Re}\left(Re^{i\theta}\right) = R\cos\theta$$

$$\mathrm{Im}\left(Re^{i\theta}\right) = R\sin\theta$$

$$\left|Re^{i\theta}\right| = |R|$$

$$\arg\left(Re^{i\theta}\right) = \mathrm{mod}\left[\mathrm{atan}\left[\frac{R\sin[\theta]}{R\cos[\theta]}\right], 2\pi\right]$$

Without i

$\forall a, b: \mathbb{R} \cdot$

$$\mathrm{first}_c[(a, b)_c] = a$$

$$\mathrm{second}_c[(a, b)_c] = b$$

$$|(a, b)_c| = \sqrt{a^2 + b^2}$$

$$\arg[(a, b)_c] = \mathrm{mod}[\mathrm{atan2}[b, a], 2\pi]$$

$\forall R, \theta: \mathbb{R} \cdot$

$$\mathrm{first}_c\left[(R, \theta)_p\right] = R\cos[\theta]$$

$$\mathrm{second}_c\left[(R, \theta)_p\right] = R\sin[\theta]$$

$$\left|(R, \theta)_p\right| = |R|$$

$$\arg\left[(R, \theta)_p\right] = \mathrm{mod}\left[\mathrm{atan2}\left[R\sin[\theta], R\cos[\theta]\right], 2\pi\right]$$

Square brackets are used in the new version for function inputs to avoid any ambiguity, but this has nothing to do with vectors or complex numbers. It is just a general clarification.

One of the features of the polar representation is that the same vector can be represented by an unlimited number of alternative number pairs. Adding integer multiples of 2π to the angle takes you to another, while reversing the sign of the radius and adjusting the angle by π takes you to yet another.

Instead of the inverse tangent function I have used the $\mathrm{atan2}$ function, which has become a frequent part of computer programming languages and has the advantage that it gives a result in the correct quadrant. (However, not necessarily in the range $[0,2\pi)$, which is why the modulus with 2π is taken.

Conversions

The same vector can be represented in Cartesian and polar forms, and there are endless alternative pairs of values for the polar form. I have defined functions to convert between forms and to return the polar version that has a positive radius and an angle in the first rotation.

With i

$\forall R, \theta : \mathbb{R} \cdot$

$$Re^{i\theta} = R\cos\theta + iR\sin\theta$$

$\forall a, b : \mathbb{R} \cdot$

$$a + ib = \sqrt{a^2 + b^2}\, e^{i\tan^{-1}\frac{b}{a}}$$

$\forall R, \theta : \mathbb{R}, k : \mathbb{Z} \cdot$

$$Re^{i\theta} = -Re^{i\theta + (2k+1)\pi}$$

$$Re^{i\theta} = Re^{i\theta + 2k\pi}$$

Without i

$\forall R, \theta : \mathbb{R} \cdot$

$$(R, \theta)_p = (R\cos[\theta], R\sin[\theta])_c$$

$\forall a, b : \mathbb{R} \cdot$

$$(a, b)_c = (\sqrt{a^2 + b^2}, \mathrm{mod}[\mathrm{atan2}[b, a], 2\pi])_p$$

$\forall R, \theta : \mathbb{R}, k : \mathbb{Z} \cdot$

$$(R, \theta)_p = (-R, \theta + (2k+1)\pi)_p$$

$$(R, \theta)_p = (R, \theta + 2k\pi)_p$$

$\mathrm{toPolar} : \mathbb{V} \to \mathbb{V}$

$\forall a, b : \mathbb{R} \cdot$

$$\mathrm{toPolar}[(a, b)_c] = (\sqrt{a^2 + b^2}, \mathrm{mod}[\mathrm{atan2}[b, a], 2\pi])_p$$

$\mathrm{toCartesian} : \mathbb{V} \to \mathbb{V}$

$\forall R, \theta : \mathbb{R} \cdot$

$$\mathrm{toCartesian}\left[(R, \theta)_p\right] = (R\cos[\theta], R\sin[\theta])_c$$

$\mathrm{principal} : \mathbb{V} \to \mathbb{V}$

$\forall R, \theta : \mathbb{R} \cdot$

$$\mathrm{principal}\left[(R, \theta)_p\right] = \mathrm{toPolar}\left[\mathrm{toCartesian}\left[(R, \theta)_p\right]\right]$$

Addition and subtraction

These are the usual operations for two dimensional vectors.

With i

$\forall z_1, z_2, z_3 : \mathbb{C} \cdot$

$$z_1 + z_2 = z_2 + z_1$$

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

$$z_1 - z_2 + z_2 = z_1$$

$\forall a_1, a_2, b_1, b_2 : \mathbb{R} \cdot$

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$$

$$(a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i(b_1 - b_2)$$

Without i

$\forall r_1, r_2, r_3 : \mathbb{V} \cdot$

$$r_1 + r_2 = r_2 + r_1$$

$$r_1 + (r_2 + r_3) = (r_1 + r_2) + r_3$$

$$r_1 - r_2 + r_2 = r_1$$

$\forall a_1, a_2, b_1, b_2 : \mathbb{R} \cdot$

$$(a_1, b_1)_c + (a_2, b_2)_c = (a_1 + a_2, b_1 + b_2)_c$$

$$(a_1, b_1)_c - (a_2, b_2)_c = (a_1 - a_2, b_1 - b_2)_c$$

Addition and subtraction using the exponential/polar form is complicated and not shown above. It involves converting to Cartesian form, doing the addition or subtraction, then converting back.

Multiplication and division

This is where the new vector operations properly begin, highlighted by circles in their symbols.

Whereas addition and subtraction are simplest when written using Cartesian form, rotating multiplication and division are easiest in the polar form. Having defined rotating multiplication, the rule for rotating division can be deduced.

With i

$\forall z_1, z_2, z_3 : \mathbb{C} \cdot$

$$z_1 z_2 = z_2 z_1$$

$$z_1(z_2 z_3) = (z_1 z_2) z_3$$

$$z_1\left(\frac{z_2}{z_3}\right) = \frac{(z_1 z_2)}{z_3}$$

$$\frac{z_1}{z_2} z_2 = z_1$$

$\forall a_1, a_2, b_1, b_2 : \mathbb{R} \cdot$

$$(a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

$$(a_1 + ib_1) \div (a_2 + ib_2) = \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right) + i\left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right)$$

$\forall R_1, R_2, \theta_1, \theta_2 : \mathbb{R} \cdot$

$$R_1 e^{i\theta_1} \times R_2 e^{i\theta_2} = R_1 R_2 e^{i(\theta_1 + \theta_2)}$$

$$R_1 e^{i\theta_1} \div R_2 e^{i\theta_2} = \frac{R_1}{R_2} e^{i(\theta_1 - \theta_2)}$$

Without i

$\forall r_1, r_2, r_3 : \mathbb{V} \cdot$

$$r_1 \odot r_2 = r_2 \odot r_1$$

$$r_1 \odot (r_2 \odot r_3) = (r_1 \odot r_2) \odot r_3$$

$$r_1 \odot (r_2 \oslash r_3) = (r_1 \odot r_2) \oslash r_3$$

$$r_1 \oslash r_2 \odot r_2 = r_1$$

$\forall a_1, a_2, b_1, b_2 : \mathbb{R} \cdot$

$$(a_1, b_1)_c \odot (a_2, b_2)_c = (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1)_c$$

$$(a_1, b_1)_c \oslash (a_2, b_2)_c = \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}, \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right)_c$$

$\forall R_1, R_2, \theta_1, \theta_2 : \mathbb{R} \cdot$

$$(R_1, \theta_1)_p \odot (R_2, \theta_2)_p = (R_1 R_2, \theta_1 + \theta_2)_p$$

$$(R_1, \theta_1)_p \oslash (R_2, \theta_2)_p = \left(\frac{R_1}{R_2}, \theta_1 - \theta_2\right)_p$$

Rotating exponents

Rotating exponents are defined so that they are consistent with the rotating multiplication. The most concise formula uses a mixture of Cartesian and polar forms.

With Real numbers there are some subtle complexities when trying to state useful identities. Positive Real numbers have two square roots, a positive and a negative one. The same can be said for any even number root. The exponent and radical notations are defined to return just the positive root in this case. For example:

$$\forall x{:}\,\mathbb{R} \mid x \geq 0 \cdot \sqrt{x} \geq 0 \wedge \sqrt{x} \times \sqrt{x} = x \wedge \sqrt{x} = x^{\frac{1}{2}}$$

Another wrinkle is that negative Real numbers do not have even roots. We cannot say, for example, that:

$$\sqrt{-1} \times \sqrt{-1} = \sqrt{-1 \times -1}$$

because the left hand side is not defined, and so cannot be equal to the right hand side. We also cannot say that:

$$(-1)^{\frac{2}{2}} = -1$$

because the initial square root of -1 does not exist.

Identities have to be qualified, like this example:

$$\forall a, b{:}\,\mathbb{R} \mid a \geq 0 \wedge b \geq 0 \cdot \sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

A more comprehensive statement of the identities available is:

$$\forall a, b{:}\,\mathbb{R}, n, k{:}\,\mathbb{Z} \mid a \geq 0 \wedge b \geq 0 \wedge k \neq 0 \cdot a^{\frac{n}{2k}} b^{\frac{n}{2k}} = (ab)^{\frac{n}{2k}}$$

$$\forall a, b{:}\,\mathbb{R}, n, k{:}\,\mathbb{Z} \cdot a^{\frac{n}{2k+1}} b^{\frac{n}{2k+1}} = (ab)^{\frac{n}{2k+1}}$$

With complex numbers and rotating vectors the problems are slightly different. All have roots, but the problem of deciding which root to use is harder. For example, there are n rotating vectors that could be the nth root of a rotating vector. For a vector $(R, \theta)_p$ with R positive and θ in $[0,2\pi)$ the set of nth roots is:

$$\left\{k{:}\,\mathbb{Z} \mid k \in 0..n-1 \cdot \left(R^{\frac{1}{n}}, \frac{\theta + k2\pi}{n}\right)_p\right\}$$

With k outside the range shown the roots are just duplicates.

The principal value, the one chosen, should be (in polar notation) the version with a positive radius and the smallest available angle in the interval $[0,2\pi)$.

Without careful qualification, this leads to problems when alternative paths to a result are taken. For example, using complex numbers:

$$\sqrt{-1} \times \sqrt{-1} = \sqrt{-1 \times -1} = \sqrt{1} = 1$$

but

$\sqrt{-1} \times \sqrt{-1} = i \times i = -1$

so the first identity (i.e. $\sqrt{-1} \times \sqrt{-1} = \sqrt{-1 \times -1}$) is not valid, even though both sides are defined.

If we consider all possible roots in the above calculations then the two results, 1 and -1, are both found depending on which roots are used.

In the table below I have gone straight to the most comprehensive statement on exponents, which is for a vector raised to the power of a vector (complex number with complex exponent). This could also be approached in steps by taking simpler cases first, such as a vector raised to a natural number, a vector raised to a negative integer, and so on.

With i

$\forall z_1, z_2, z_3, z_4 : \mathbb{C} \cdot$

$$\left(z_1^{z_3}\right)\left(z_2^{z_4}\right) = z_1^{z_3} z_2^{z_4}$$

$$z_1^{z_2} z_1^{z_3} = z_1^{z_2+z_3}$$

$$\frac{z_1^{z_2}}{z_1^{z_3}} = z_1^{z_2-z_3}$$

$$\left(z_1^{z_3}\right)^{z_3} = z_1^{z_2 z_3}$$

$\forall R, \theta, a, b : \mathbb{R} \cdot$

$$(Re^{i\theta})^{a+ib} = R^a e^{-b\theta} e^{i(b \ln R + \quad)}$$

$\forall a, b, c, d : \mathbb{R} \cdot$

$$(a + ib)^{c+id} = \rho^c e^{-d\theta} (\cos[c\theta + d \ln[\rho]] + i \ \sin[c\theta + d \ln[\rho]])$$

where $\rho = \sqrt{a^2 + b^2}$

$\theta = \tan^{-1}\left[\frac{b}{a}\right]$ with θ in the correct quadrant

Without i

$\forall r_1, r_2, r_3, r_4 : \mathbb{V} \cdot$

$$(r_1 \circledast r_3) \odot (r_2 \circledast r_4) = r_1 \circledast r_3 \odot r_2 \circledast r_4$$

$$(r_1 \circledast r_2) \odot (r_1 \circledast r_3) = r_1 \circledast (r_2 + r_3)$$

$$(r_1 \circledast r_2) \oslash (r_1 \circledast r_3) = r_1 \circledast (r_2 - r_3)$$

$$(r_1 \circledast r_2) \circledast r_3 = r_1 \circledast (r_2 \odot r_3)$$

$\forall R, \theta, a, b : \mathbb{R} \cdot$

$$(R, \theta)_p \circledast (a, b)_c = \left(R^a e^{-b} \quad , b \ln[R] + a\theta\right)_p$$

$\forall a, b, c, d : \mathbb{R} \cdot$

$$(a, b)_c \circledast (c, d)_c = \left(\rho^c e^{-d\theta}, c\theta + d \ln[\rho]\right)_p$$

where $\rho = \sqrt{a^2 + b^2}$

$\theta = \text{atan2}[b, a]$

With this definition, the familiar addition of powers rule works with two dimensional vectors too.

$\forall R, \theta, a_1, b_1, a_2, b_2 : \mathbb{R} \cdot$

$$(R,\theta)_p \circledast (a_1,b_1)_c \odot (R,\theta)_p \circledast (a_2,b_2)_c$$

$$= \left(R^{a_1}e^{-b_1\theta}, b_1\ln[R] + a_1\theta\right)_p \odot \left(R^{a_2}e^{-b_2\theta}, b_2\ln[R] + a_2\theta\right)_p$$

$$= \left(R^{a_1}e^{-b_1\theta}R^{a_2}e^{-b_2\theta}, b_1\ln[R] + a_1\theta + b_2\ln[R] + a_2\theta\right)_p$$

$$= \left(R^{a_1+a_2}e^{-(b_1+b_2)\theta}, (b_1+b_2)\ln[R] + (a_1+a_2)\theta\right)_p$$

$$= (R,\theta)_p \circledast (a_1+a_2, b_1+b_2)_c$$

$$= (R,\theta)_p \circledast ((a_1,b_1)_c + (a_2,b_2)_c)$$

The multiplication of powers rule also works with this definition of rotating powers:

$\forall R, \theta, a_1, b_1, a_2, b_2 : \mathbb{R} \cdot$

$$(R,\theta)_p \circledast (a_1,b_1)_c \circledast (a_2,b_2)_c$$

$$= \left(R^{a_1}e^{-b_1\theta}, b_1\ln[R] + a_1\theta\right)_p \circledast (a_2,b_2)_c$$

$$= \left(\left(R^{a_1}e^{-b_1\theta}\right)^{a_2}e^{-b_2(b_1\ln[R]+a_1\theta)}, b_2\ln\left[R^{a_1}e^{-b_1\theta}\right] + a_2(b_1\ln[R] + a_1\theta)\right)_p$$

$$= \left(R^{a_1a_2}e^{-a_2b_1\theta}e^{-b_1b_2\ln[R]}e^{-a_1b_2\theta}, b_2\ln[R^{a_1}] + b_2\ln\left[e^{-b_1\theta}\right] + a_2b_1\ln[R] + a_1a_2\theta\right)_p$$

$$= \left(R^{a_1a_2}e^{-a_1b_2\theta-a_2b_1\theta}R^{-b_1b_2}, a_1b_2\ln[R] - b_1b_2\theta + a_2b_1\ln[R] + a_1a_2\theta\right)_p$$

$$= \left(R^{a_1a_2-b_1b_2}e^{-(a_1b_2+a_2b_1)\theta}, (a_1b_2+a_2b_1)\ln[R] + (a_1a_2-b_1b_2)\theta\right)_p$$

$$= (R,\theta)_p \circledast (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1)_c$$

$$= (R,\theta)_p \circledast ((a_1,b_1)_c \odot (a_2,b_2)_c)$$

Here are some special cases of exponents.

With i

$\forall a, b, c : \mathbb{R} \cdot$

$$
\begin{aligned}
& a^{b+i} \\
&= e^{(\ln(a))(b+ic)} \\
&= e^{b\ln(a)+ic\ln(a)} \\
&= e^{b\,\mathrm{l}\ \ (a)} e^{ic\ln(a)} \\
&= a^b e^{ic\ln(a)} \\
&= e^{b\ln(a)}(\cos c\ln(a) + i \sin c\ln(a)) \\
&= a^b(\cos c\ln(a) + i \sin c\ln(a)) \\
&= a^b \cos c\ln(a) + i a^b \sin c\ln(a)
\end{aligned}
$$

$\forall \theta : \mathbb{R} \cdot$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$\forall a : \mathbb{R} \cdot$

$$e^a = e^a$$

$\forall a, b : \mathbb{R} \cdot$

$$e^{a+ib} = e^a e^{ib}$$

Without i

$\forall a, b, c : \mathbb{R} \cdot$

$$
\begin{aligned}
& a \circledast (b, c)_c \\
&= e \circledast (\ln[a] \odot (b, c)_c) \\
&= e \circledast (b\ln[a], c\ln[a])_c \\
&= e^{b\ln[a]} \odot (1, c\ln[a])_p \\
&= \left(a^b, c\ln[a]\right)_p \\
&= e^{b\ln(a)} \odot (\cos[c\ln[a]], \sin[c\ln[\mathrm{a}]])_c \\
&= a^b \odot (\cos[c\ln[a]], \sin[c\ln[\mathrm{a}]])_c \\
&= \left(a^b \cos[c\ln[a]], a^b \sin[c\ln[\mathrm{a}]]\right)_c
\end{aligned}
$$

$\forall \theta : \mathbb{R} \cdot$

$$(e, 0)_p \circledast (0, \theta)_c = (1, \theta)_p$$

$\forall a : \mathbb{R} \cdot$

$$(e, 0)_p \circledast (a, 0)_c = (e^a, 0)_p$$

$\forall a, b : \mathbb{R} \cdot$

$$(e, 0)_p \circledast (a, b)_c = (e, 0)_p \circledast (a, 0)_c \odot (e, 0)_p \circledast (0, b)_c = (e^a, b)_p$$

Rotating logarithms

Rotating logs are defined consistently with powers and multiplication. The most complicated case is the log of a two dimensional vector to a two dimensional base.

These numbers (like their complex counterparts) have no obvious geometric meaning. However, it is still possible to convert to logs, do operations, then convert back, in order to reach results that might have been harder otherwise. This is the way that ordinary logarithms have often been used.

With complex numbers and rotating vectors there is an unlimited set of alternative logarithms so a principal value has to be returned by the logarithm functions.

With i

$\forall z_1, z_2, z_3 : \mathbb{C} \cdot$

$$z_1^{\,z_2} = z_3 \Rightarrow \log_{z_1} z_3 = z_2$$

$$\log_z z_1 + \log_z z_2 = \log_z z_1 z_2$$

$$\log_z z_1 - \log_z z_2 = \log_z \frac{z_1}{z_2}$$

$$z_2 \log_z z_1 = \log_z z_1^{\,z_2}$$

$\forall R_1, \theta_1, R_2, \theta_2 : \mathbb{R} \cdot$

$$\log_{R_1 e^{i\theta_1}}\left(R_2 e^{i\theta_2}\right) = \frac{\theta_1\theta_2 + \ln(R_1)\ln(R_2)}{\theta_1^2 + \ln(R_1)^2} + i\,\frac{\theta_2\ln(R_1) - \theta_1\ln(R_2)}{\theta_1^2 + \ln(R_1)^2}$$

$\forall R, \theta : \mathbb{R} \cdot$

$$\ln\left(Re^{i\theta}\right) = \ln R + i\theta$$

Without i

$\forall r_1, r_2, r_3 : \mathbb{V} \cdot$

$$(r_1 \circledast r_2) = r_3 \Rightarrow \log_{\circ r_1}[r_3] = r_2$$

$$\log_{\circ r}[r_1] + \log_{\circ r}[r_2] = \log_{\circ r}[r_1 \odot r_2]$$

$$\log_{\circ r}[r_1] - \log_{\circ r}[r_2] = \log_{\circ r}[r_1 \oslash r_2]$$

$$r_2 \odot \log_{\circ r}[r_1] = \log_{\circ r}[r_1 \circledast r_2]$$

$\forall R_1, \theta_1, R_2, \theta_2 : \mathbb{R} \cdot$

$$\log_{\circ (R_1,\theta_1)_p}\left[(R_2,\theta_2)_p\right] = \left(\frac{\theta_1\theta_2 + \ln[R_1]\ln[R_2]}{\theta_1^2 + \ln[R_1]^2}, \frac{\theta_2\ln[R_1] - \theta_1\ln[R_2]}{\theta_1^2 + \ln[R_1]^2}\right)_c$$

$\forall R, \theta : \mathbb{R} \cdot$

$$\ln_\circ\left[(R,\theta)_p\right] = (\ln[R], \theta)_c$$

The rotating log formula was deduced from the rotating power formula as follows:

$$(R_1,\theta_1)_p \circledast (a,b)_c = \left(R_1^a e^{-b\theta_1}, b\ln[R_1] + a\theta_1\right)_p = (R_2,\theta_2)_p$$

$$R_2 = R_1^a e^{-b\theta_1}$$

$$\ln[R_2] = \ln\left[R_1^a e^{-b\theta_1}\right]$$

$$\ln[R_2] = a\ln[R_1] - b\theta_1$$

$$b = \frac{a\ln[R_1] - \ln[R_2]}{\theta_1}$$

$$\theta_2 = b\ln[R_1] + a\theta_1$$

$$\theta_2 = \left(\frac{a\ln[R_1] - \ln[R_2]}{\theta_1}\right)\ln[R_1] + a\theta_1$$

$$\theta_1\theta_2 = aln[R_1]^2 - \ln[R_1]\ln[R_2] + a\theta_1^2$$

$$\theta_1\theta_2 + \ln[R_1]\ln[R_2] = aln[R_1]^2 + a\theta_1^2$$

$$a = \frac{\theta_1\theta_2 + \ln[R_1]\ln[R_2]}{\theta_1^2 + \ln[R_1]^2}$$

$$\theta_2 = b\ln[R_1] + \left(\frac{\theta_1\theta_2 + \ln[R_1]\ln[R_2]}{\theta_1^2 + \ln[R_1]^2}\right)\theta_1$$

$$b\ln[R_1] = \theta_2 - \left(\frac{\theta_1\theta_2 + \ln[R_1]\ln[R_2]}{\theta_1^2 + \ln[R_1]^2}\right)\theta_1$$

$$b\ln[R_1] = -\left(\frac{\theta_2\theta_1^2 + \theta_2\ln[R_1]^2 - \theta_1^2\theta_2 - \theta_1\ln[R_1]\ln[R_2]}{\theta_1^2 + \ln[R_1]^2}\right)$$

$$b = \frac{\theta_2\ln[R_1] - \theta_1\ln[R_2]}{\theta_1^2 + \ln[R_1]^2}$$

$$\log_{\circ(R_1,\theta_1)_p}\left[(R_2,\theta_2)_p\right] = \left(\frac{\theta_1\theta_2 + \ln[R_1]\ln[R_2]}{\theta_1^2 + \ln[R_1]^2}, \frac{\theta_2\ln[R_1] - \theta_1\ln[R_2]}{\theta_1^2 + \ln[R_1]^2}\right)_c$$

Rotating logs can be added, subtracted, multiplied, and divided just like ordinary logs. First, adding two logs is like multiplying the numbers they are logs of.

$\forall R_1, \theta_1, R_2, \theta_2, R_3, \theta_3 : \mathbb{R} \cdot$

$$\log_{\circ(R_1,\theta_1)_p}\left[(R_2,\theta_2)_p\right] + \log_{\circ(R_1,\theta_1)_p}\left[(R_3,\theta_3)_p\right]$$

$$= \left(\frac{\theta_1\theta_2 + \ln[R_1]\ln[R_2]}{\theta_1^2 + \ln[R_1]^2}, \frac{\theta_2\ln[R_1] - \theta_1\ln[R_2]}{\theta_1^2 + \ln[R_1]^2}\right)_c + \left(\frac{\theta_1\theta_3 + \ln[R_1]\ln[R_3]}{\theta_1^2 + \ln[R_1]^2}, \frac{\theta_3\ln[R_1] - \theta_1\ln[R_3]}{\theta_1^2 + \ln[R_1]^2}\right)_c$$

$$= \left(\frac{\theta_1\theta_2 + \ln[R_1]\ln[R_2] + \theta_1\theta_3 + \ln[R_1]\ln[R_3]}{\theta_1^2 + \ln[R_1]^2}, \frac{\theta_2\ln[R_1] - \theta_1\ln[R_2] + \theta_3\ln[R_1] - \theta_1\ln[R_3]}{\theta_1^2 + \ln[R_1]^2}\right)_c$$

$$= \left(\frac{\theta_1(\theta_2 + \theta_3) + \ln[R_1](\ln[R_2] + \ln[R_3])}{\theta_1^2 + \ln[R_1]^2}, \frac{(\theta_2 + \theta_3)\ln[R_1] - \theta_1(\ln[R_2] - \ln[R_3])}{\theta_1^2 + \ln[R_1]^2}\right)_c$$

$$= \left(\frac{\theta_1(\theta_2 + \theta_3) + \ln[R_1]\ln[R_2R_3]}{\theta_1^2 + \ln[R_1]^2}, \frac{(\theta_2 + \theta_3)\ln[R_1] - \theta_1\ln[R_2R_3]}{\theta_1^2 + \ln[R_1]^2}\right)_c$$

$$= \log_{\circ(R_1,\theta_1)_p}\left[(R_2,\theta_2)_p \odot (R_3,\theta_3)_p\right]$$

Subtracting logs works too.

$\forall R_1, \theta_1, R_2, \theta_2, R_3, \theta_3 : \mathbb{R} \cdot$

$$\log_{\circ(R_1,\theta_1)_p}\left[(R_2,\theta_2)_p\right] - \log_{\circ(R_1,\theta_1)_p}\left[(R_3,\theta_3)_p\right]$$

$$= \left(\frac{\theta_1\theta_2 + \ln[R_1]\ln[R_2]}{\theta_1^2 + \ln[R_1]^2}, \frac{\theta_2\ln[R_1] - \theta_1\ln[R_2]}{\theta_1^2 + \ln[R_1]^2}\right)_c - \left(\frac{\theta_1\theta_3 + \ln[R_1]\ln[R_3]}{\theta_1^2 + \ln[R_1]^2}, \frac{\theta_3\ln[R_1] - \theta_1\ln[R_3]}{\theta_1^2 + \ln[R_1]^2}\right)_c$$

$$= \left(\frac{\theta_1\theta_2 + \ln[R_1]\ln[R_2] - \theta_1\theta_3 - \ln[R_1]\ln[R_3]}{\theta_1^2 + \ln[R_1]^2}, \frac{\theta_2\ln[R_1] - \theta_1\ln[R_2] - \theta_3\ln[R_1] + \theta_1\ln[R_3]}{\theta_1^2 + \ln[R_1]^2}\right)_c$$

$$= \left(\frac{\theta_1(\theta_2 - \theta_3) + \ln[R_1](\ln[R_2] - \ln[R_3])}{\theta_1^2 + \ln[R_1]^2}, \frac{(\theta_2 - \theta_3)\ln[R_1] - \theta_1(\ln[R_2] - \ln[R_3])}{\theta_1^2 + \ln[R_1]^2}\right)_c$$

$$= \left(\frac{\theta_1(\theta_2 - \theta_3) + \ln[R_1]\ln\left[\frac{R_2}{R_3}\right]}{\theta_1^2 + \ln[R_1]^2}, \frac{(\theta_2 - \theta_3)\ln[R_1] - \theta_1\ln\left[\frac{R_2}{R_3}\right]}{\theta_1^2 + \ln[R_1]^2}\right)_c$$

$$= \log_{\circ(R_1,\theta_1)_p}\left[(R_2,\theta_2)_p \oslash (R_3,\theta_3)_p\right]$$

And multiplication:

$\forall R_1, \theta_1, R_2, \theta_2, a, b: \mathbb{R} \cdot$

$$(a,b)_c \odot \log_{\circ(R_1,\theta_1)_p}\left[(R_2,\theta_2)_p\right]$$

$$= (a,b)_c \odot \left(\frac{\theta_1\theta_2 + \ln[R_1]\ln[R_2]}{\theta_1^2 + \ln[R_1]^2}, \frac{\theta_2\ln[R_1] - \theta_1\ln[R_2]}{\theta_1^2 + \ln[R_1]^2}\right)_c$$

$$= \left(\frac{a(\theta_1\theta_2 + \ln[R_1]\ln[R_2]) - b(\theta_2\ln[R_1] - \theta_1\ln[R_2])}{\theta_1^2 + \ln[R_1]^2}, \frac{a(\theta_2\ln[R_1] - \theta_1\ln[R_2]) + b(\theta_1\theta_2 + \ln[R_1]\ln[R_2])}{\theta_1^2 + \ln[R_1]^2}\right)_c$$

$$= \left(\frac{\theta_1 b\ln[R_2] + a\theta_1\theta_2 + a\ln[R_1]\ln[R_2] - b\theta_2\ln[R_1]}{\theta_1^2 + \ln[R_1]^2}, \frac{b\ln[R_2]\ln[R_1] + a\theta_2\ln[R_1] - a\theta_1\ln[R_2] + b\theta_1\theta_2}{\theta_1^2 + \ln[R_1]^2}\right)_c$$

$$= \left(\frac{\theta_1(b\ln[R_2] + a\theta_2) + \ln[R_1]\ln\left[R_2^a e^{-b\theta_2}\right]}{\theta_1^2 + \ln[R_1]^2}, \frac{(b\ln[R_2] + a\theta_2)\ln[R_1] - \theta_1\ln\left[R_2^a e^{-b\theta_2}\right]}{\theta_1^2 + \ln[R_1]^2}\right)_c$$

$$= \log_{\circ(R_1,\theta_1)_p}\left[\left(R_2^a e^{-b\theta_2}, b\ln[R_2] + a\theta_2\right)_p\right]$$

$$= \log_{\circ(R_1,\theta_1)_p}\left[(R_2,\theta_2)_p \circledast (a,b)_c\right]$$

Scalar operations

Arguably, these are unnecessary since one could use a vector instead, but with zero for the second value.

With i

$\forall a, b, c : \mathbb{R} \cdot$

$$c(a + ib) = ca + icb$$

$$\frac{a + ib}{c} = \frac{a}{c} + i\frac{b}{c}$$

$$c + (a + ib) = (c + a) + ib$$

$$(a + ib) - c = (a - c) + ib$$

$$a^{b+ic} = a^b e^{ic\ln(a)}$$

$$a^{b+ic} = a^b \cos c\ln(a) + ia^b \sin c\ln(a)$$

$\forall a, R, \theta : \mathbb{R} \cdot$

$$a\left(Re^{i\theta}\right) = aRe^{i\theta}$$

$$\frac{Re^{i\theta}}{a} = \frac{R}{a}e^{i\theta}$$

$$(Re^{i\theta})^n = R^n e^{in\theta}$$

Without i

$\forall a, b, c : \mathbb{R} \cdot$

$$c(a, b)_c = (ca, cb)_c$$

$$\frac{(a, b)_c}{c} = \left(\frac{a}{c}, \frac{b}{c}\right)_c$$

$$c + (a, b)_c = (a + c, b)_c$$

$$(a, b)_c - c = (a - c, b)_c$$

$$a \circledast (b, c)_c = \left(a^b, c\ln[a]\right)_p$$

$$a \circledast (b, c)_c = \left(a^b \cos[c\ln[a]], a^b \sin[c\ln[a]]\right)_c$$

$$e \circledast (b, c)_c = \left(e^b, c\right)_p$$

$$e \circledast (0, c)_c = (1, c)_p$$

$\forall a, R, \theta : \mathbb{R} \cdot$

$$a(R, \theta)_p = (aR, \theta)_p$$

$$\frac{(R, \theta)_p}{a} = \left(\frac{R}{a}, \theta\right)_p$$

$$(R, \theta)_p \circledast n = (R^n, n\theta)_p$$

Trigonometric functions

Once again, a rotating version can be defined analogous to the complex definition.

With i

$\forall a, b : \mathbb{R} \cdot$

$$\cos[a + ib] = \cos[a]\left(\frac{e^b + e^{-b}}{2}\right) + i\sin[a]\left(\frac{e^{-b} - e^b}{2}\right)$$

$$\sin[a + ib] = \sin[a]\left(\frac{e^b + e^{-b}}{2}\right) - i\cos[a]\left(\frac{e^{-b} - e^b}{2}\right)$$

$$\cosh[a + ib] = \cosh[a]\cos[b] + i\sinh[a]\sin[b]$$

$$\sinh[a + ib] = \sinh[a]\cos[b] + i\cosh[a]\sin[b]$$

$\forall z : \mathbb{C} \cdot$

$$\cos^{-1} z = -i\ln\left[z \pm i\sqrt{1 - z^2}\right]$$

$$\sin^{-1} z = -i\ln\left[iz \pm \sqrt{1 - z^2}\right]$$

$$\tan^{-1} z = \frac{i}{2}\ln\left[\frac{i + z}{i - z}\right]$$

$\forall \theta : \mathbb{R} \cdot$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{i2}$$

Without i

$\forall a, b : \mathbb{R} \cdot$

$$\cos_\circ[(a, b)_c] = \left(\cos[a]\left(\frac{e^b + e^{-b}}{2}\right), \sin[a]\left(\frac{e^{-b} - e^b}{2}\right)\right)_c$$

$$\sin_\circ[(a, b)_c] = \left(\sin[a]\left(\frac{e^b + e^{-b}}{2}\right), -\cos[a]\left(\frac{e^{-b} - e^b}{2}\right)\right)_c$$

$$\cosh_\circ[(a, b)_c] = (\cosh[a]\cos[b], \sinh[a]\sin[b])_c$$

$$\sinh_\circ[(a, b)_c] = (\sinh[a]\cos[b], \cosh[a]\sin[b])_c$$

$\forall r : \mathbb{V} \cdot$

$$\mathrm{acos}_\circ[r] = (0, -1)_c \odot \log_{\circ e}\left[r \pm (0,1)_c \odot (1 - r \circledast 2) \circledast \left(\frac{1}{2}\right)\right]$$

$$\mathrm{asin}_\circ[r] = (0, -1)_c \odot \log_{\circ e}\left[(0,1)_c \odot r \pm (1 - r \circledast 2) \circledast \left(\frac{1}{2}\right)\right]$$

$$\mathrm{atan}_\circ[r] = (0,1)_c \oslash 2 \odot \log_{\circ e}[((0,1)_c + r) \oslash ((0,1)_c - r)]$$

$\forall \theta : \mathbb{R} \cdot$

$$(\cos[\theta], 0)_c = (\cos[\theta], 0)_p = \left((1, \theta)_p + (1, -\theta)_p\right) \oslash (2,0)_c$$

$$(\sin[\theta], 0)_c = (\sin[\theta], 0)_p = \left((1, \theta)_p - (1, -\theta)_p\right) \oslash (0,2)_c$$

Differentiation

A function taking a two dimensional vector as input and returning a two dimensional vector could, in principle, give four derivatives. There is an established notation for this. For example, a function from (x, y) to (u, v) could give the matrix of derivatives represented by:

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

The same could be said, in principle, for a function that maps between complex numbers. However, complex derivatives have been defined so that the functions that can be differentiated are only those where the derivative is the same from all directions. The rule for differentiability is summarised as the Cauchy-Reimann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

This restriction means that the derivative of a differentiable function of a complex variable will be another function of a complex variable. It also means that there is upward compatibility with derivatives of Real variables.

With i

$\forall f: \mathbb{C} \to \mathbb{C}, z, z_0: \mathbb{C}, x, y: \mathbb{R}, u, v: (\mathbb{R} \times \mathbb{R}) \to \mathbb{R} \cdot$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$f(z) = u(x, y) + iv(x, y) \;\Rightarrow$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Without i

$\forall f: \mathbb{V} \to \mathbb{V}, r, r_0: \mathbb{V}, x, y: \mathbb{R}, u, v: (\mathbb{R} \times \mathbb{R}) \to \mathbb{R} \cdot$

$$f^{\circ}[r_0] = \lim_{r \to r_0} (f[r] - f[r_0]) \oslash (r - r_0)$$

$$f[r] = (u[x, y], v[x, y])_c \;\Rightarrow$$

$$f^{\circ}[r] = \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)_c = \left(\frac{\partial v}{\partial y}, -\frac{\partial u}{\partial y}\right)_c$$

$\forall f: \mathbb{C} \rightarrow \mathbb{C}, z: \mathbb{C}, n, a, b, c, d: \mathbb{R} \cdot$

$$f(z) = z^n \;\Rightarrow\; f'(x) = nz^{n-1}$$

$$f(z) = \frac{az+b}{cz+d}, \;\Rightarrow\; f'(x) = \frac{ad-bc}{(cz+d)^2}$$

$$f(z) = \ln(z) \;\Rightarrow\; f'(x) = \frac{1}{z}$$

$$f(z) = \exp(az), \;\Rightarrow\; f'(x) = a\exp(az)$$

$\forall f: \mathbb{V} \rightarrow \mathbb{V}, r: \mathbb{V}, n, a, b, c, d: \mathbb{R} \cdot$

$$f[r] = r \circledast n \;\Rightarrow\; f^{\circ}[r] = n \odot r \circledast (n-1)$$

$$f[r] = \frac{(a \odot r \oplus b)}{(c \odot r \oplus d)} \;\Rightarrow\; f^{\circ}[r] = (ad-bc) \oslash (c \odot r + d) \circledast 2$$

$$f[r] = \ln_{\circ}[r] \;\Rightarrow\; f^{\circ}[r] = 1 \oslash r$$

$$f[r] = \exp_{\circ}[a \odot r] \;\Rightarrow\; f^{\circ}[r] = a \odot \exp_{\circ}[a \odot r]$$

$\forall f, g: \mathbb{C} \rightarrow \mathbb{C}, z: \mathbb{C} \cdot$

$$\big(f(z)g(z)\big)' = f'(z)g(z) + f(z)g'(z)$$

$$(f(z) + g(z))' = f'(z) + g'(z)$$

$$(f(g(z)))' = f'(g(z))g'(z)$$

$$(cf(z))' = cf'(z), \qquad c: \mathbb{R} \text{ is a constant}$$

$$\left(\frac{1}{f(z)}\right)' = \frac{-1}{f(z)^2} f'(z)$$

$\forall f, g: \mathbb{V} \rightarrow \mathbb{V}, r: \mathbb{V} \cdot$

$$(f[r] \odot g[r])^{\circ} = f^{\circ}[r] \odot g[r] + f[r] \odot g^{\circ}[r]$$

$$(f[r] + g[r])^{\circ} = f^{\circ}[r] + g^{\circ}[r]$$

$$\big(f[g[r]]\big)^{\circ} = f^{\circ}[g[r]] \odot g^{\circ}[r]$$

$$(c \odot f[r])^{\circ} = c \odot f^{\circ}[r], \qquad c: \mathbb{R} \text{ is a constant}$$

$$(1 \oslash f[r])^{\circ} = (-1 \oslash f[r] \circledast 2) \odot f^{\circ}[r]$$

Conjugates

In complex number theory, complex conjugates are often referred to.

With i

$\forall a, b: \mathbb{R} \cdot$

$$(a+ib)^* = a - ib$$

$$(a+ib)(a-ib) = a^2 + b^2$$

$$(a,b)_c^{**} = (a,b)_c$$

Without i

$\forall a, b: \mathbb{R} \cdot$

$$(a,b)_c^{*} = (a,-b)_c$$

$$(a,b)_c \odot (a,-b)_c = (a^2+b^2, 0)_c$$

$$(a,b)_c^{**} = (a,b)_c$$

Unit vectors

Another common topic.

With i

$i \times i = -1$

$e^{i\frac{\pi}{2}} \times e^{i\frac{\pi}{2}} = e^{i\pi}$

$e^{i\theta} = \cos\theta + i\ \sin\theta$

$e^{in\theta} = \cos n\theta + i\ \sin n\theta$

$e^{-in\theta} = \cos n\theta - i\ \sin n\theta$

$e^{in\theta} + e^{-in\theta} = 2\cos n\theta$

$e^{in\theta} - e^{-in\theta} = 2i \sin n\theta$

Without i

$(0,1)_c \odot (0,1)_c = (-1,0)_c$

$\left(1,\frac{\pi}{2}\right)_p \odot \left(1,\frac{\pi}{2}\right)_p = (1,\pi)_p$

$(1,\theta)_p = (\cos[\theta], \sin[\theta])_c$

$(1,n\theta)_p = (\cos[n\theta], \sin[n\theta])_c$

$(1,-n\theta)_p = (\cos[n\theta], -\sin[n\theta])_c$

$(1,n\theta)_p + (1,-n\theta)_p = (2\cos[n\theta], 0)_c$

$(1,n\theta)_p - (1,-n\theta)_p = (0, 2\sin[n\theta])_c$

Three dimensions

Rotating vector operations in three dimensions are possible, but the formula for Cartesian rotating multiplication is quite lengthy.

$(a,b,c)_c$

$(R,\theta,\varphi)_p$

$\forall R,\theta,\varphi{:}\ \mathbb{R} \cdot$

$$(R,\theta,\varphi)_p = (R\sin[\theta]\cos[\varphi], R\sin[\theta]\sin[\varphi], R\cos[\theta])_c$$

$\forall a, b, c\colon \mathbb{R} \cdot$

$$(a, b, c)_c = (\sqrt{a^2 + b^2 + c^2}, \text{acos}\left[\frac{c}{\sqrt{a^2 + b^2 + c^2}}\right], \text{mod}\left[\text{atan2}[b, a]\right], 2\pi)_p$$

$\forall R_1, \theta_1, \varphi_1, R_2, \theta_2, \varphi_2\colon \mathbb{R} \cdot$

$$(R_1, \theta_1, \varphi_1)_p \odot (R_2, \theta_2, \varphi_2)_p = (R_1 R_2, \theta_1 + \theta_2, \varphi_1 + \varphi_2)_p$$

$\forall a_1, b_1, c_1, a_2, b_2, c_2\colon \mathbb{R} \cdot$

$$(a_1, b_1, c_1)_c \odot (a_2, b_2, c_2)_c = \left((a_1 a_2 - b_1 b_2)\left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}} + \frac{c_2}{\sqrt{a_2^2 + b_2^2}}\right), (a_1 b_2 + a_2 b_1)\left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}} + \frac{c_2}{\sqrt{a_2^2 + b_2^2}}\right), c_1 c_2 - \sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}\right)_c$$

The lengthy formula for multiplication of vectors in the Cartesian form can be derived as follows:

If

$(R_1, \theta_1, \varphi_1)_p = (a_1, b_1, c_1)_c$ and $(R_2, \theta_2, \varphi_2)_p = (a_2, b_2, c_2)_c$

and

$(R_1, \theta_1, \varphi_1)_p \odot (R_2, \theta_2, \varphi_2)_p = (R_1 R_2, \theta_1 + \theta_2, \varphi_1 + \varphi_2)_p$

$(R_1, \theta_1, \varphi_1)_p \odot (R_2, \theta_2, \varphi_2)_p = (a_3, b_3, c_3)_c$

then

$a_3 = R_1 R_2 \sin[\theta_1 + \theta_2] \cos[\varphi_1 + \varphi_2]$

$= R_1 R_2 (\sin[\theta_1] \cos[\theta_2] + \sin[\theta_2] \cos[\theta_1])(\cos[\varphi_1] \cos[\varphi_2] - \sin[\varphi_1] \sin[\varphi_2])$

$= R_1 R_2 (\sin[\theta_1] \cos[\theta_2] \cos[\varphi_1] \cos[\varphi_2] - \sin[\theta_1] \cos[\theta_2] \sin[\varphi_1] \sin[\varphi_2] + \sin[\theta_2] \cos[\theta_1] \cos[\varphi_1] \cos[\varphi_2] - \sin[\theta_2] \cos[\theta_1] \sin[\varphi_1] \sin[\varphi_2])$

$= R_1 \sin[\theta_1] \cos[\varphi_1] R_2 \cos[\theta_2] \cos[\varphi_2] - R_1 \sin[\theta_1] \sin[\varphi_1] R_2 \cos[\theta_2] \sin[\varphi_2] + R_2 \sin[\theta_2] \cos[\varphi_2] R_1 \cos[\theta_1] \cos[\varphi_1] - R_2 \sin[\theta_2] \sin[\varphi_2] R_1 \cos[\theta_1] \sin[\varphi_1]$

$= a_1 c_2 \cos[\varphi_2] - b_1 c_2 \sin[\varphi_2] + a_2 c_1 \cos[\varphi_1] - b_2 c_1 \sin[\varphi_1]$

$$= a_1 c_2 \frac{a_2}{\sqrt{a_2^2+b_2^2}} - b_1 c_2 \frac{b_2}{\sqrt{a_2^2+b_2^2}} + a_2 c_1 \frac{a_1}{\sqrt{a_1^2+b_1^2}} - b_2 c_1 \frac{b_1}{\sqrt{a_1^2+b_1^2}}$$

$$= \frac{c_2}{\sqrt{a_2^2+b_2^2}}(a_1 a_2 - b_1 b_2) + \frac{c_1}{\sqrt{a_1^2+b_1^2}}(a_1 a_2 - b_1 b_2)$$

$$= (a_1 a_2 - b_1 b_2)\left(\frac{c_1}{\sqrt{a_1^2+b_1^2}} + \frac{c_2}{\sqrt{a_2^2+b_2^2}}\right)$$

$$b_3 = R_1 R_2 \sin[\theta_1+\theta_2]\sin[\varphi_1+\varphi_2]$$

$$= R_1 R_2 (\sin[\theta_1]\cos[\theta_2] + \sin[\theta_2]\cos[\theta_1])(\sin[\varphi_1]\cos[\varphi_2] + \sin[\varphi_2]\cos[\varphi_1])$$

$$= R_1 R_2 (\sin[\theta_1]\cos[\theta_2]\sin[\varphi_1]\cos[\varphi_2] + \sin[\theta_1]\cos[\theta_2]\sin[\varphi_2]\cos[\varphi_1] + \sin[\theta_2]\cos[\theta_1]\sin[\varphi_1]\cos[\varphi_2] + \sin[\theta_2]\cos[\theta_1]\sin[\varphi_2]\cos[\varphi_1])$$

$$= R_1 \sin[\theta_1]\sin[\varphi_1] R_2 \cos[\theta_2]\cos[\varphi_2] + R_1 \sin[\theta_1] R_2 \cos[\theta_2]\sin[\varphi_2]\cos[\varphi_1] + R_2 \sin[\theta_2]\cos[\varphi_2] R_1 \sin[\varphi_1]\cos[\theta_1] + R_2 \sin[\theta_2]\sin[\varphi_2] R_1 \cos[\theta_1]\cos[\varphi_1]$$

$$= b_1 c_2 \cos[\varphi_2] + a_1 c_2 \sin[\varphi_2] + a_2 c_1 \sin[\varphi_1] + b_2 c_1 \cos[\varphi_1]$$

$$= b_1 c_2 \frac{a_2}{\sqrt{a_2^2+b_2^2}} + a_1 c_2 \frac{b_2}{\sqrt{a_2^2+b_2^2}} + a_2 c_1 \frac{b_1}{\sqrt{a_1^2+b_1^2}} + b_2 c_1 \frac{a_1}{\sqrt{a_1^2+b_1^2}}$$

$$= \frac{c_2}{\sqrt{a_2^2+b_2^2}}(a_2 b_1 + a_1 b_2) + \frac{c_1}{\sqrt{a_1^2+b_1^2}}(a_2 b_1 + a_1 b_2)$$

$$= (a_1 b_2 + a_2 b_1)\left(\frac{c_1}{\sqrt{a_1^2+b_1^2}} + \frac{c_2}{\sqrt{a_2^2+b_2^2}}\right)$$

$$c_3 = R_1 R_2 \cos[\theta_1+\theta_2]$$

$$= R_1 R_2 (\cos[\theta_1]\cos[\theta_2] - \sin[\theta_1]\sin[\theta_2])$$

$$= R_1 \cos[\theta_1] R_2 \cos[\theta_2] - R_1 \sin[\theta_1] R_2 \sin[\theta_2]$$

$$= c_1 c_2 - \sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}$$

So, finally:

$$(a_1, b_1, c_1)_c \odot (a_2, b_2, c_2)_c = \left((a_1 a_2 - b_1 b_2)\left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}} + \frac{c_2}{\sqrt{a_2^2 + b_2^2}}\right), (a_1 b_2 + a_2 b_1)\left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}} + \frac{c_2}{\sqrt{a_2^2 + b_2^2}}\right), c_1 c_2 - \sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2} \right)_c$$

Appendix B: Illustrations of rotating vector operations with trigonometric identities

Stephenson's book, 'Mathematical methods for science students' (second edition) includes demonstrations of using complex numbers to establish complicated trigonometrical identities. Here are two of those translated into the form of vectors with rotating operations.

From page 119.

$(\cos[6\theta], \sin[6\theta])_c$

$= (\cos[\theta], \sin[\theta])_c \circledast 6$

$= ((\cos[\theta], 0)_c + (0, \sin[\theta])_c) \circledast 6$

Binomial expansion

$= (\cos[\theta], 0)_c \circledast 6$

$+((\cos[\theta], 0)_c \circledast 5 \odot (0, \sin[\theta])_c) \odot 6$

$+((\cos[\theta], 0)_c \circledast 4 \odot (0, \sin[\theta])_c \circledast 2) \odot 15$

$+((\cos[\theta], 0)_c \circledast 3 \odot (0, \sin[\theta])_c \circledast 3) \odot 20$

$+((\cos[\theta], 0)_c \circledast 2 \odot (0, \sin[\theta])_c \circledast 4) \odot 15$

$+((\cos[\theta], 0)_c \odot (0, \sin[\theta])_c \circledast 5) \odot 6$

$+(0, \sin[\theta])_c \circledast 6$

Powers of $(0, a)_c$ are used in the next step. They are not obvious and were worked out separately. See below.

$= (\cos[\theta]^6, 0)_c$

$+((\cos[\theta]^5, 0)_c \odot (0, \sin[\theta])_c) \odot 6$

$+((\cos[\theta]^4, 0)_c \odot (-\sin[\theta]^2, 0)_c) \odot 15$

$+((\cos[\theta]^3, 0)_c \odot (0, -\sin[\theta]^3)_c) \odot 20$

$+((\cos[\theta]^2, 0)_c \odot (\sin[\theta]^4, 0)_c) \odot 15$

$+((\cos[\theta], 0)_c \odot (0, \sin[\theta]^5)_c) \odot 6$

$+(-\sin[\theta]^6, 0)_c$

$= (\cos[\theta]^6, 0)_c$

$+(0, \cos[\theta]^5 \sin[\theta])_c \odot 6$

$+(-\cos[\theta]^4 \sin[\theta]^2, 0)_c \odot 15$

$+(0, -\cos[\theta]^3 \sin[\theta]^3)_c \odot 20$

$+(\cos[\theta]^2 \sin[\theta]^4, 0)_c \odot 15$

$+(0, \cos[\theta] \sin[\theta]^5)_c \odot 6$

$+(-\sin[\theta]^6, 0)_c$

$$= (\cos[\theta]^6 - 15\cos[\theta]^4 \sin[\theta]^2 + 15\cos[\theta]^2 \sin[\theta]^4 - \sin[\theta]^6, 6\cos[\theta]^5 \sin[\theta] - 20\cos[\theta]^3 \sin[\theta]^3 + 6\cos[\theta] \sin[\theta]^5)_c$$

So, the final conclusion is that:

$\cos[6\theta] = \cos[\theta]^6 - 15\cos[\theta]^4 \sin[\theta]^2 + 15\cos[\theta]^2 \sin[\theta]^4 - \sin[\theta]^6$

$\sin[6\theta] = 6\cos[\theta]^5 \sin[\theta] - 20\cos[\theta]^3 \sin[\theta]^3 + 6\cos[\theta] \sin[\theta]^5$

This follows because, at every stage, we have been writing about the same vector.

Powers of $(0, a)_c$ used above, are shown now:

$(0, a)_c \circledast 2 = (0, a)_c \odot (0, a)_c = (-a^2, 0)_c$

$(0, a)_c \circledast 3 = (-a^2, 0)_c \odot (0, a)_c = (0, -a^3)_c$

$(0, a)_c \circledast 4 = (0, -a^3)_c \odot (0, a)_c = (a^4, 0)_c$

$(0, a)_c \circledast 5 = (a^4, 0)_c \odot (0, a)_c = (0, a^5)_c$

$(0, a)_c \circledast 6 = (0, a^5)_c \odot (0, a)_c = (-a^6, 0)_c$

$(0, a)_c \circledast 7 = (-a^6, 0)_c \odot (0, a)_c = (0, -a^7)_c$

$(0, a)_c \circledast 8 = (0, -a^7)_c \odot (0, a)_c = (a^8, 0)_c$

$(0, a)_c \circledast 9 = (a^8, 0)_c \odot (0, a)_c = (0, a^9)_c$

From page 118. This begins by establishing some identities to use later:

$z = (\cos[\theta], \sin[\theta])_c = (1, \theta)_p$

$\frac{1}{z} = (\cos[\theta], -\sin[\theta])_c = (1, -\theta)_p$

$z + \frac{1}{z} = (\cos[\theta], \sin[\theta])_c + (\cos[\theta], -\sin[\theta])_c = (2\cos[\theta], 0)_c$

$z^n = (\cos[\theta], \sin[\theta])_c \circledast n = (1, \theta)_p \circledast n = (1, n\theta)_p = (\cos[n\theta], \sin[n\theta])_c$

$\frac{1}{z^n} = (\cos[\theta], -\sin[\theta])_c \circledast -n = (1, -\theta)_p \circledast -n = (1, -n\theta)_p = (\cos[n\theta], -\sin[n\theta])_c$

$z^n + \frac{1}{z^n} = (\cos[n\theta], \sin[n\theta])_c + (\cos[n\theta], -\sin[n\theta])_c = (2\cos[n\theta], 0)_c$

These identities are then used to break down a high powered trig function:

$(2^6 \cos[\theta]^6, 0)_c$

$= (2\cos[\theta], 0)_c \circledast 6$

$= \left((1, \theta)_p + (1, -\theta)_p\right) \circledast 6$

Now a binomial expansion:

$= (1, \theta)_p \circledast 6$

$+\left((1, \theta)_p \circledast 5 \odot (1, -\theta)_p \circledast 1\right) \odot 6$

$+\left((1, \theta)_p \circledast 4 \odot (1, -\theta)_p \circledast 2\right) \odot 15$

$+\left((1, \theta)_p \circledast 3 \odot (1, -\theta)_p \circledast 3\right) \odot 20$

$+\left((1, \theta)_p \circledast 2 \odot (1, -\theta)_p \circledast 4\right) \odot 15$

$+\left((1, \theta)_p \circledast 1 \odot (1, -\theta)_p \circledast 5\right) \odot 6$

$+(1, -\theta)_p \circledast 6$

$= (1,6\theta)_p$

$+\big((1,5\theta)_p \odot (1,-\theta)_p\big) \odot 6$

$+\big((1,4\theta)_p \odot (1,-2\theta)_p\big) \odot 15$

$+\big((1,3\theta)_p \odot (1,-3\theta)_p\big) \odot 20$

$+\big((1,2\theta)_p \odot (1,-4\theta)_p\big) \odot 15$

$+\big((1,\theta)_p \odot (1,-5\theta)_p\big) \odot 6$

$+(1,-6\theta)_p$

$= (1,6\theta)_p$

$+(1,4\theta)_p \odot 6$

$+(1,2\theta)_p \odot 15$

$+(1,0)_p \odot 20$

$+(1,-2\theta)_p \odot 15$

$+(1,-4\theta)_p \odot 6$

$+(1,-6\theta)_p$

$= \big((1,6\theta)_p + (1,-6\theta)_p\big) + \big((1,4\theta)_p + (1,-4\theta)_p\big) \odot 6 + \big((1,2\theta)_p + (1,-2\theta)_p\big) \odot 15 + (1,0)_p \odot 20$

$= (2\cos[6\theta], 0)_c + (2\cos[4\theta], 0)_c \odot 6 + (2\cos[2\theta], 0)_c \odot 15 + (20,0)_p$

$= (2\cos[6\theta], 0)_c + (12\cos[4\theta], 0)_c + (30\cos[2\theta], 0)_c + (20,0)_p$

$= (2\cos[6\theta] + 12\cos[4\theta] + 30\cos[2\theta] + 20\,, 0)_c$

At every stage this is the same vector, so we can infer that, as claimed earlier:

$2^6\cos[\theta]^6 = 2\cos[6\theta] + 12\cos[4\theta] + 30\cos[2\theta] + 20$

Appendix C: Solution of cubic equations

This was the original inspiration for complex numbers. It can also be written using vectors with rotating operations.

$ax^3 + bx^2 + cx + d = 0, \qquad a \neq 0$

Restyled as:

$x: \mathbb{V}, a, b, c, d: \mathbb{R} \mid a \neq 0 \cdot$

$$a \odot x \circledast 3 + b \odot x \circledast 2 + c \odot x + d = (0,0)_c$$

Calculate the discriminant and some other intermediate values:

$D = 18abcd - 4b^3d + b^2d^2 - 4ac^3 - 27a^2d^2$

$A = b^2 - 3ac$

$B = 2b^3 - 9abc + 27a^2d$

$$C = \sqrt[3]{\frac{B \pm \sqrt{B^2 - 4A^3}}{2}}$$

or

$$C = \sqrt[3]{\frac{B \pm \sqrt{27a^2D}}{2}}$$

Either result for C may be chosen unless $A = 0$, in which case plus or minus must be chosen so that the terms do not cancel each other out. Roots are as follows:

$$k \in \{0,1,2\} \cdot x_k = -\frac{1}{3a}\left(b + \left(\left(1, \frac{2\pi}{3}\right)_p \circledast k\right) \odot C + \frac{A}{\left(\left(1, \frac{2\pi}{3}\right)_p \circledast k\right) \odot C}\right)$$

Which gives a root for each value of k, as follows:

$$x_0 = \left(-\frac{1}{3a}\left(b + C + \frac{A}{C}\right), 0\right)_p$$

$$x_1 = -\frac{1}{3a}\left(b + \left(C, \frac{2\pi}{3}\right)_p + \frac{A}{\left(C, \frac{2\pi}{3}\right)_p}\right)$$

$$x_2 = -\frac{1}{3a}\left(b + \left(C, \frac{4\pi}{3}\right)_p + \frac{A}{\left(C, \frac{4\pi}{3}\right)_p}\right)$$